

The Theory of Transmission Functions and its Application to Design Verification of Large Scale Safety Systems

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ABSTRACT: In a prior paper [1] we described some of the problems posed by the optimization of adequate protections and its verification. We showed that the main problem lies within the realm of finding envelopes of the evolution of damage indicators in piecewise sequences of changes in dynamics associated to protection actions and/or the onset of stochastic phenomena (cascade effects).

We started by the analysis of linear piecewise sequences, that are able to describe a large group of situations, and explored the concept of Transmission Functions (Funciones de Transmisión, FT) of Multi-Input Multi-Output (MIMO) linear piecewise systems. The FT concept was defined for all piecewise linear MIMO, but the transmission functions were only found in case of sequences of Canonical Single-Input Single-Output (CSISO, Phase variable/Frobenius) systems, showing the difficulties of its extension to the general case.

In this paper we generalize the use of these techniques by reducing the general MIMO case to the canonical case. It solves the problem of discontinuous derivatives at piece boundaries, a feature essential to protection problems, as the protective actions try precisely to sharply change the wrong trends of the system evolution. The results substantially differ from the canonical cases (continuous derivatives), of course coming down to them in this very special case. Conclusions are given about the efficiency of this method as compared to other possibilities, and a discussion is made of its implications in the context of Protection Engineering (i.e., application of reliability engineering to system safety).

Keywords: Protection Engineering; System Safety; Piecewise Linear MIMO Systems; System Dynamics; Probabilistic Safety Assessment; Dynamic Reliability

I. INTRODUCTION. SCOPE AND OBJECTIVES.

In a prior paper [1] we described some of the problems posed by the optimization of adequate protections and its verification and defined some key concepts of industrial facilities protection engineering, like damage domains. We showed that the main problem lies within the realm of finding envelopes of the evolution of damage indicators in piecewise sequences of changes in dynamics, (sequence of transitions, SOT), associated to protection actions and/or the onset of stochastic phenomena (cascade effects).

In general, the problem focuses in ensuring that the envelopes consider all the possible SOT that may be involved in the accident progression and then cover, for each sequence, uncertainty in initial conditions and key data, as well as, most important, boundary conditions and protective actions timing. These uncertainties easily explode the number of situations to consider, so that brute force techniques based on reproducing transients with best estimate simulations usually fail in the completeness of the situations considered to demonstrate the envelope character.

In order to generate new techniques for their study, we started with linear piecewise sequences that are able to describe a large group of situations. In reference [1] we explored the concept of Transmission Functions (Funciones de Transmisión, FT) of sequences of MIMO linear piecewise systems. The FT concept was defined for all linear Multi-Input Multi-Output (MIMO) systems, but they were only found in case of sequences of CSISO canonical systems (Canonical Single-Input Single-Output), showing the difficulties of its extension to the general case. In this paper we show how to circumvent these difficulties and extend the results to the general MIMO piecewise linear case by making a new application of the well-known canonical phase form transform (Frobenius) theorem [2] applied together with properties of piecewise linear systems and their piece transfer functions. The mathematical treatment is self-contained and do not require the background of [1], being consistent with it after considering minor changes in notations.

MIMO transmission functions extend the concepts, properties and computations of transfer functions to piecewise linear SOT, including equivalent features like automatic rules for network topologies and configurations (series, parallel, as well as feedback loops; see [4]). They may be associated to each of the most protection-relevant variables, (namely the protection stimuli variables, -see [3], [5]-, i.e., exceeding certain thresholds on them would open the possibility of some damage). They are able to effectively compute the basic impact of the protective-degrading actions on the stimuli variables for the myriad of combinations resulting from the possible action times, as well as for the different possible boundary conditions that isolate the plant/overall system models from the outside world (thus generating a large variety of possible time input shapes).

MIMO FTs help then the optimization of the process to identify sequence failure domains¹, i.e. the set of SOTs that exceed safety limits. They also facilitate the calculations of the aggregate contribution of its elements to the sequence frequency [3], [5], that is, the frequency of exceeding safety limits, main figure of merit in protection engineering [6]. Section 6.2 further elaborates this. This contribution extends the canonical case results of [1] to the more general and realistic case of any type of piecewise-linear dynamics and solves the important problem of derivative discontinuities that characterize dynamic events (i.e., events inducing dynamic transitions).

The paper is organized as follows:

Section 2 is devoted to the presentation of the problem in a classical mathematical framework and to describe an alternate approach based on the reduction of the general MIMO problem to the canonical case. After introducing the partition operators math frame (section 2.1), the solution is restated in terms of the transfer functions corresponding to all the input output pairs (FT matrix approach, section 2.2), which leads to the definition of the Transmission Functions FTs and provides a first, however inefficient, method to compute them. Sections 2.3 and 2.4 describe then the reduction of the evolution of a given process vector component under MIMO piecewise linear sequences, to the evolution of its canonical variables (Theorem 1) via the well known canonical phase form transform (Frobenius) theorem. In section 2.5 we exemplify this to compute the first two intervals of a SOT. Section 3 presents our alternate approach. In section 3.1 the transmission functions for process variables are expressed in terms of recurrence relations among them. Its equations are the main result of this paper. Section 3.2 provides the proof. Section 3.3 checks over the proposal through two significant applications to the continuous case of reference [1] showing its consistency and to the pure linear case, i.e., when all the transfer functions are the same, then independent on the interval.

Section 4 summarizes the final results of the theory. These are verified via a set of examples in section 5. Section 5.1 includes a simple detailed analytical example to further facilitate understanding. Section 5.2 compares the results with the FT matrix approach of section 2.2 using MATLAB in several significant examples of varied complexity. Finally, in section 6 we summarize conclusions and make a discussion of the findings with due comments about the usefulness of this approach. Sections 7 and 8 provide adequate references and a glossary of the notations. Appendix A considers transfer functions in partitions of the time span, defines their notations and properties illustrating the techniques with significant examples via pole decomposition, including details of the analytical example of section 5.1.

II. BASIC APPROACH.

This section is devoted to describe the straightforward matrix solution approach for general piecewise linear MIMO systems written in terms of a matrix product of all the components of the transfer functions of each single piece, as well as to state the basic elements of our alternate approach. Assume that $x_j(t)$ describes the time evolve of the process variable x_j , i.e., component j of the N th order process vector \bar{x} , during a time interval where an event has occurred at time T_n , inducing a transition in the dynamics that sharply bends the trajectory followed by the system before the occurrence of the event $0 < t < T_n$. Components j are assumed to define a state vector, i.e., its knowledge at time 0 allows to know its future evolution. We also assume the evolution governed by a piecewise-controllable linear MIMO of M pieces, defined in the $n+1$ -th piece ($1 \leq n+1 \leq M$) by

$$\begin{aligned} \dot{x}_j^{n+1}(\tau_{n+1}) &= \sum_{q=1}^N [A_{n+1}]_{jq} x_q^{n+1}(\tau_{n+1}) + \sum_{i=1}^N [B_{n+1}]_{ji} u_{n+1,i}(\tau_{n+1}) \quad 0 < T_n + \tau_{n+1} \leq T_{n+1} \\ \bar{u}_{n+1}(\tau_{n+1}) &= \begin{cases} u_{n+1,i}(\tau_{n+1}) & i = 1, \dots, L \\ 0 & i = L+1, \dots, N \end{cases} \end{aligned} \quad [1]$$

with $[B_{n+1}]$ an $N \times N$ matrix. Vector $\bar{u}_{n+1}(\tau_{n+1})$ defines L analytic input signals (i.e., functions of time, continuous and with continuous derivatives within each piece). Vector \bar{x} is assumed continuous in the whole time span $0 < T_{M-1} + \tau_M$ and known at $t = 0$.

¹ In the context of this paper failure domains and damage domains are the same.

The solution² is easily reduced to the case $[B_{n+1}] = [I_N]$. In addition, the superposition principle allows us to find the solution as an additive set

$$x_j^{n+1}(t) = \sum_{i=1}^L \sum_{k=1}^N [B_{n+1}]_{jk} x_k^{n+1,i}(t) \quad [2]$$

of solutions of the L simpler problems in each interval

$$x_j^{n+1,i}(\tau_{n+1}) = \sum_{q=1}^N [A_{n+1}^{eq}]_{jq} x_q^{n+1,i}(\tau_{n+1}) + \delta_{j,i} u_{n+1,i}(\tau_{n+1}) ; [A_{n+1}^{eq}] = [B_{n+1}]^{-1} [A_{n+1}] [B_{n+1}] \quad [3]$$

each corresponding to the single input $\delta_{j,i} u_{n+1,i}(\tau_{n+1})$.

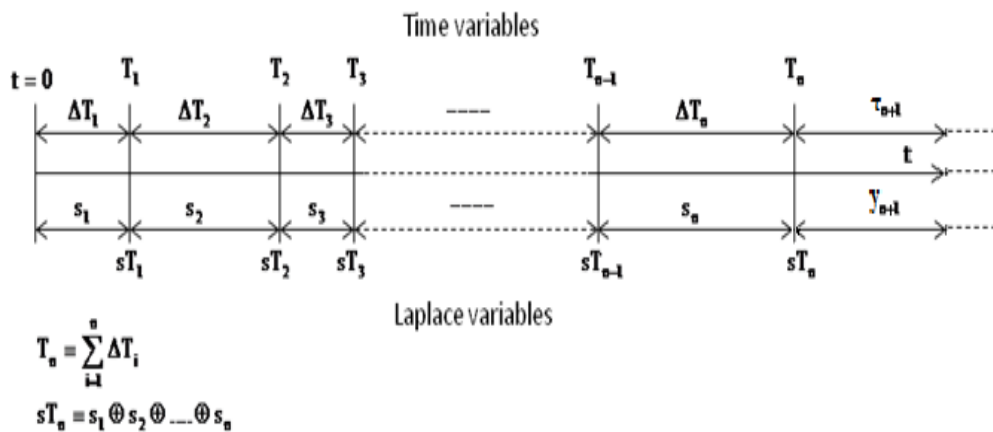


Figure 1. Time diagram and notation.

Assuming the contribution of every input i to the output component j a continuous function in the time span, then the boundary conditions are

$$\sum_{k=1}^N [B_{n+1}]_{jk} x_k^{n+1,i}(\tau_{n+1} = 0) = \sum_{k=1}^N [B_n]_{jk} x_k^{n,i}(\tau_n = T_n - T_{n-1} = \Delta T_n) \quad [4]$$

for transition from piece n to piece n+1 (see figure 1) at time T_n , whereas initial conditions $x_j^{1,i}(0)$ are known.

Without loss of generality, in the rest of this section and in section 3, we concentrate on finding the contribution of each of the L inputs, i.e., the solution of equation [3] and drop the superscript “eq” from the A matrices, considering $[B_{n+1}] = [I_N]$, so equation [4] becomes

$$x_j^{n+1,i}(\tau_{n+1} = 0) = x_j^{n,i}(\tau_n = \Delta T_n) \quad [5]$$

Section 4 below will compile the results in the general case.

2.1. Mathematical framework. Partition operators of functions and polynomials.

First, concerning notations, symbol $\tilde{\varphi}(s)$ means the Laplace transform of function $\varphi(t)$, and symbol $\tilde{\varphi}(s_1, s_2, \dots)$ that of function $\varphi(t_1, t_2, \dots)$ of the independent variables t_1, t_2, \dots . In general, we will be dealing with cases where these time variables constitute a partition in n intervals (see figure 1). The associated Laplace

² We assume that matrix B has an inverse. Note that the apparent complexity of the reduction to L problems is the same as in standard linear systems of a single interval.

variables are denoted $s_{n,m} \equiv (s_n, \dots, s_m)$. With $T_{n,m} = \sum_{j=m}^n t_j$, this aggregate interval has a single, associated Laplace Transform variable $sT_{n,m}$.

If $\varphi(t)$ is continuous and with continuous derivatives

$$\tilde{\varphi}(sT_{2,1}) \equiv \tilde{\varphi}(s_1 \oplus s_2) = \frac{\tilde{\varphi}(s_1) - \tilde{\varphi}(s_2)}{s_2 - s_1} \quad [6]$$

is the Laplace transform of function $\varphi(t)$, being $t = t_1 + t_2$, when considered as a function of the aggregate time of the two variables t_1 and t_2 or separately for each one. That is, the \oplus operator partitions variable $\varphi(t)$ in two intervals. The notation $\tilde{\varphi}(sT_{n,m}) = \tilde{\varphi}(s_n \oplus s_{n-1} \oplus \dots \oplus s_m)$ is used to indicate the general piecewise case $\tilde{\varphi}(s_{n,m})$ in the particular case of φ continuous and with all derivatives continuous at the intervals boundaries³.

The following property holds

$$[\tilde{\varphi}_1(s)\tilde{\varphi}_2(s)]_{s=s_1 \oplus s_2} = \tilde{\varphi}_1(s_1)\tilde{\varphi}_2(s_1 \oplus s_2) + \tilde{\varphi}_1(s_1 \oplus s_2)\tilde{\varphi}_2(s_2) \quad [7]$$

that may be iterated for partitions including more intervals and their variables,

$$\tilde{\varphi}_1(sT_{n,m}) \otimes \tilde{\varphi}_2(sT_{n,m}) \equiv [\tilde{\varphi}_1(s)\tilde{\varphi}_2(s)]_{s=sT_{n,m}} = \sum_{k=m}^n \tilde{\varphi}_1(sT_{n,k})\tilde{\varphi}_2(sT_{k,m}) \quad [8]$$

These operations are representative of the interaction between different intervals when they are partitions of a larger one, and are valid provided the functions are continuous and with continuous derivatives there. However, care should be exercised in using them, as the symbols are not associated with the addition and product usual operations and do not have their properties. Because they are unfamiliar to many in the engineering community, in appendix A some important properties of partitions of transfer functions are given, including the generalization and explicit expressions of these operations in terms of more familiar ones in significant examples, like the classical pole decomposition.

As is the case with transfer functions in general, it is convenient to consider them as operators acting on functions (as for instance the input functions), which gives math sense to polynomials in s as combinations of Dirac distributions (time pulses) and its derivatives, strictly well defined via classical generalized functions (see reference [7]), and common place in classical control theory.

2.2. Transmission Functions of process variables. The FT matrix approach.

Due to the continuity of vector \bar{x} between intervals as stated in equation [5], the solution of system [3] (see figure 1) may be written in terms of its multiple Laplace transform $\tilde{x}_j^{n+1,i}(y_{n+1}, s_n, s_{n-1}, \dots, s_1)$ where variables s_l ($l = 1, \dots, n$) are the Laplace variables that describe the partition of the time span into n intervals

$$\begin{aligned} t &= \tau_{n+1} + \Delta T_n + \dots + \Delta T_1 \\ 0 < t_l &< \Delta T_l \equiv T_l - T_{l-1} \quad l = 1, \dots, n \end{aligned} \quad [9]$$

y_{n+1} representing the one associated to τ_{n+1} , $0 < \tau_{n+1} \leq \Delta T_{n+1}$ and s_l that of ΔT_l . This solution reads, in terms of the well known transfer functions matrix, $[G^n(s_n)]$ of each interval, whose elements are defined as⁴

³ Note that $sT_{n,m}$ is a single variable while $s_{n,m}$ denotes the set of local variables in the partition

⁴ Note that this relationship establish a strong correlation among the components of matrix $[G^n(s_n)]$. Only those polynomials Q generated by a single matrix A are acceptable.

$$G_{j \leftarrow i}^n(s_n) \equiv [s_n [I] - [A_n]]_{ji}^{-1} \equiv \frac{Q_{j \leftarrow i}^n(s_n)}{P^n(s_n)}$$

$$P^n(s) = s^N + \sum_{k=0}^{N-1} p_{n,k} s^k$$

$$Q_{j \leftarrow i}^n(s) = \sum_{k=0}^{N-1} Q_{jk}^{n,i} s^k$$
[10]

as follows

$$\tilde{x}_j^{n+1,i}(y_{n+1}, s_n, \dots, s_1) = G_{j \leftarrow i}^{n+1}(y_{n+1}) \tilde{u}_{n+1,i}(y_{n+1} \oplus sT_n) + \sum_{k_n=1}^N G_{j \leftarrow k_n}^{n+1}(y_{n+1}) \tilde{x}_{k_n}^{n,i}(s_n, s_{n-1}, \dots, s_1)$$
[11]

Unraveling [11] for n intervals, assuming the input continuous and with all derivatives continuous⁵:

$$\tilde{x}_j^{n+1,i}(y_{n+1}, s_{n,1}) \equiv \tilde{x}_j^{n+1,i}(y_{n+1}, s_n, \dots, s_1) = \sum_{m=1}^{n+1} FT_{n+1,m}^{j,i}(y_{n+1}, s_{n,m}) \tilde{u}_{m,i}(sT_m)$$

$$\begin{cases} FT_{n+1,m}^{j,i}(y_{n+1}, s_{n,m}) \equiv \sum_{k_n, \dots, k_m=1}^N G_{j \leftarrow k_n}^{n+1}(y_{n+1}) G_{k_n \leftarrow k_{n-1}}^n(s_n) \dots G_{k_{m+1} \leftarrow k_m}^{m+1}(s_{m+1}) G_{k_m \leftarrow i}^m(s_m) \\ FT_{n+1,n+1}^{j,i}(y_{n+1}) \equiv G_{j \leftarrow i}^{n+1}(y_{n+1}) \end{cases}$$
[12]

The equations in [12] define the Transmission Functions FTs and provide a first method to calculate them (FT matrix approach). It is clear that these functions represent the influence of the rest of the system on a given output component j at interval n+1, and the contribution to it of input i at a given partition interval m (see figure 2). As a result, they are useful in those piecewise problems where only a few output components are looked for, as it is the case of the stimuli variables of protective actions.

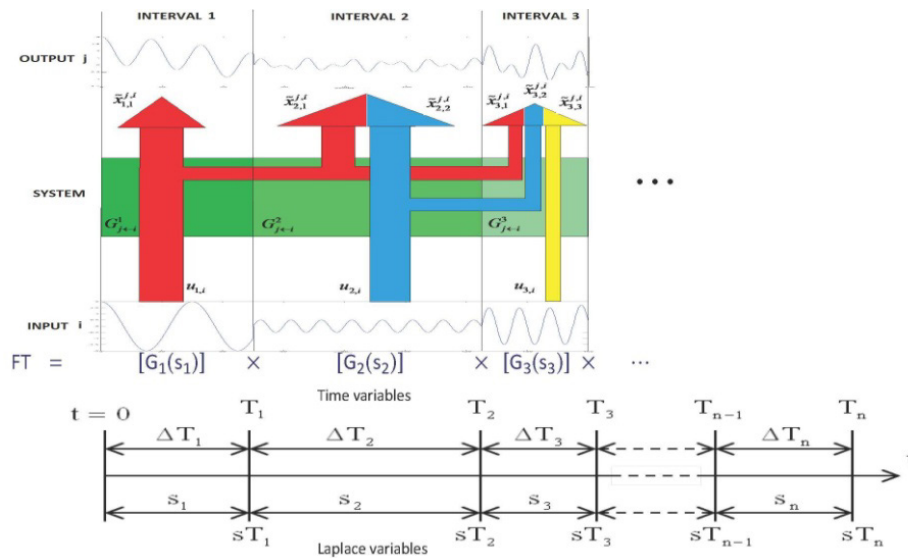


Figure 2. FTs as contribution weights of interval m, input i to a given output j.

However, finding these functions via computing the above expression is inefficient for several intervals and high order matrices $[A_n]$, except for very specific $[A_n]$, as we showed in reference [1] for canonical matrices. Fortunately (see footnote 3), the transfer function matrix components involved in the $G_{k_{m+1} \leftarrow k_m}^{m+1}(s_{m+1})$ matrices are related between themselves via equation [10], a fact that we indicate by using the arrows in its notation. In appendix A we describe some of its consequences, that justify specific properties of transfer

⁵ If this is not the case, we may always consider its discontinuities by dividing it in pieces increasing the number of intervals, the extra pieces with no change in their associated transfer functions.

function matrices that will be relevant in what follows like the key equation [A4]. This makes possible our alternate treatment.

2.3. Reduction of MIMO systems to canonical SISO. The discontinuity problem.

Canonical systems.

In the context of this paper, canonical piecewise systems are defined as the particular cases of MIMO linear piecewise systems described by the differential equations [3] and continuity equations [5] in the special case

$$\begin{cases} \dot{x}_j^{n,i}(\tau_n) = x_{j+1}^{n,i}(\tau_n) & j = 1, \dots, N-1 \\ \dot{x}_N^{n,i}(\tau_n) = -\sum_{k=1}^N p_{n,k-1} x_k^{n,i}(\tau_n) + u_{n,i}(\tau_n) & j = N \end{cases} \quad [13]$$

Given its special form, continuity of vector \bar{x} between intervals implies in this unique case that all derivatives of the solutions are continuous at the transition between pieces, if the inputs have this feature. On the other hand, they are a very special case of matrices $[A_n]$, i.e.,

$$[A_n] \equiv \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 1 \\ -p_{n,0} & \dots & -p_{n,N-2} & -p_{n,N-1} \end{bmatrix} \quad [14]$$

Thus, canonical systems (also called Frobenius canonical forms and Canonical Single-Input Single-Output, CSISO, here as in reference [1]) can not represent drastic changes of derivatives. However, this is an essential feature of any protective action that precisely tries to sharply change the bad trends acquired by the system.

The following paragraphs introduce the ingredients of an alternate approach to the FT function matrix described in 2.2. The essence is to apply the well known canonical phase form (Frobenius) decomposition theorem (see for instance reference [2]) to the case of piecewise partitions of the time span, such that within each one of the pieces, say n , $[A]$ is a constant matrix, i.e., $[A_n]$. The major advantage of it is that, contrary to the matrix approach, only the transfer function matrix components i, j , that is, the pre-established input and output components, are involved.

2.4. Theorem 1.

Given the piecewise-controllable linear system defined by equations [3] with continuity equations [5] and overall known initial conditions $x_j^{1,i}(0)$, there exists a piecewise linear transformation, $Q_{j,k}^{n,i}$

$$\begin{aligned} x_j^{n,i}(t) &= \sum_{k=0}^{N-1} Q_{j,k}^{n,i} z_k^{n,i}(t) \\ \tilde{x}_j^{n,i}(s) &= \sum_{k=0}^{N-1} Q_{j,k}^{n,i} \tilde{z}_k^{n,i}(s) = \sum_{k=0}^{N-1} Q_{j,k}^{n,i} s^k \tilde{z}_0^{n,i}(s) = Q_{j \leftarrow i}^n(s) \tilde{z}_0^{n,i}(s) \end{aligned} \quad [15]$$

that reduces the system in variables \bar{x} to a piecewise CSISO Frobenius, canonical form, in variables \bar{z} . The transformation and CSISO parameters within piece n , are given by matrix $[Q^{n,i}]$ that is defined via equation [10]. However, the CSISO boundary conditions between pieces are such that the derivatives of \bar{z} are discontinuous, and the discontinuity is such that

$$\begin{aligned} z_k^{n+1,i}(T_n) &= \frac{d^k z_0^{n+1,i}(T_n)}{dt^k} = \sum_{m=0}^{N-1} R_{km}^{n+1/2,i} \frac{d^m z_0^{n,i}(T_n)}{dt^m} = \sum_{m=0}^{N-1} R_{km}^{n+1/2,i} z_m^{n,i}(T_n) \\ R_{jk}^{n+1/2,i} &\equiv \sum_{m=0}^{N-1} [Q^{n+1,i}]_{jm}^{-1} [Q^{n,i}]_{mk} \quad 0 \leq k \leq N-1 \end{aligned} \quad [16]$$

where matrix $[Q^{n,i}]$ is such that

$$\sum_{k=0}^{N-1} \left\{ [Q^{n+1,l}]_{jk} z_k^{n+1}(T_n) - [Q^{n,l}]_{jk} z_k^n(T_n) \right\} = 0 \quad [17]$$

As proof we just mention, as well known, that for a time invariant system, i.e., all matrices $[A_n]$ are the same, the Frobenius transform matrix $[Q^{n,i}]$ may be easily found from the transfer functions, as the matrix elements $[Q^{n,i}]_{j,k}$ are the coefficients k of the polynomial $Q_{j \leftarrow i}^n(s)$ as indicated in equation [10]. This means that the same conclusion should be valid within any of the pieces. However, the link between pieces (boundary conditions) should be decided. Indeed, we need to distinguish the left side (n+1) and right side (n) of the derivatives at T_n . But continuity of \bar{x} equation [17] forces

$$\sum_{k=0}^{N-1} \left\{ [Q^{n+1,l}]_{jk} z_k^{n+1}(T_n) - [Q^{n,l}]_{jk} z_k^n(T_n) \right\} = 0 \Rightarrow z_j^{n+1}(T_n) = R_{jk}^{n+1/2,i} z_k^n(T_n) \quad [18]$$

which justifies equation [16]. In section 5.2 below, we also include classical matrix methods for computing the \bar{x} and \bar{z} variables evolution using this time format of the theorem, that also serve the purpose of checking it.

2.5. Theorem 1 for the first two intervals in terms of Laplace variables.

As indicated at the end of section 2.2, there are strong relationships between the transfer functions matrix components that reflect in many of the properties of appendix A, where the notation is clarified. We show below, in the case of 2 intervals, that, when combined with theorem 1, they allow an alternate treatment involving only the i, j input-output pair.

On the first interval we know that

$$\tilde{x}_j^{1,i}(y_1) = G_{j \leftarrow i}^1(y_1) \tilde{u}_{1,i}(y_1) \quad [19]$$

Let's consider the matrix approach equation [11] in the recurrence form

$$\begin{cases} \tilde{x}_j^{n+1,i}(y_{n+1}, s_{n,1}) = G_{j \leftarrow i}^{n+1}(y_{n+1}) \tilde{u}_{n+1,i}(y_{n+1} \oplus s_{n,1}) + \sum_{q=1}^N G_{j \leftarrow q}^{n+1}(y_{n+1}) \tilde{x}_q^{n,i}(s_{n,1}) \\ \tilde{x}_j^{1,i}(y_1) = G_{j \leftarrow i}^1(y_1) \tilde{u}_{1,i}(y_1) \end{cases} \quad [20]$$

Theorem 1 also allows us to write for the second interval, assuming the input continuous and with all derivatives continuous:

$$\begin{aligned} \tilde{x}_j^{2,i}(y_2, s_1) &= G_{j \leftarrow i}^2(y_2) \tilde{u}_{2,i}(y_2 \oplus s_1) + \sum_{q=1}^N G_{j \leftarrow q}^2(y_2) \sum_{p=0}^{N-1} Q_{q,p}^{1,i} [y_1^p z_0^{1,i}(y_1)]_{y_1=s_1} = \\ &= G_{j \leftarrow i}^2(y_2) \tilde{u}_{2,i}(y_2 \oplus s_1) + \sum_{q=1}^N G_{j \leftarrow q}^2(y_2) \sum_{k=0}^{N-1} Q_{q,k}^{1,i} R_k^{1+1/2,i}(s_1) z_0^{1,i}(s_1) \end{aligned} \quad [21]$$

Considering property [A7], i.e.,

$$\sum_{q=1}^N G_{j \leftarrow q}^2(y_2) Q_{q,k}^{2,i} = Q_{j \leftarrow i}^2(y_2) Z_{j \leftarrow i}^{1+1/2}(y_2 \oplus k) = Q_{j \leftarrow i}^2(y_2) \left[\frac{Q_{j \leftarrow i}^2(y_2 \oplus k)}{Q_{j \leftarrow i}^2(y_2)} - \frac{P^2(y_2 \oplus k)}{P^2(y_2)} \right] \quad [22]$$

we have

$$\begin{aligned} \tilde{x}_j^{2,i}(y_2, s_1) &= \frac{Q_{j \leftarrow i}^2(y_2)}{P^2(y_2)} \tilde{u}_{2,i}(y_2 \oplus s_1) + \\ &+ Q_{j \leftarrow i}^2(y_2) \sum_{k=0}^{N-1} \left[\frac{Q_{j \leftarrow i}^2(y_2 \oplus k)}{Q_{j \leftarrow i}^2(y_2)} - \frac{P^2(y_2 \oplus k)}{P^2(y_2)} \right] R_k^{1+1/2,i}(s_1) z_0^{1,i}(s_1) \end{aligned} \quad [23]$$

III. TRANSMISSION FUNCTIONS FOR MIMO PIECEWISE SYSTEMS AND ITS RECURRENCE RELATIONS.

3.1. Alternate computation of transmission functions of process variables. FT recurrence relations approach.

We can rewrite equation [12] in a slightly different way:

$$\begin{aligned}\tilde{x}_j^{n+1,i}(y_{n+1}, s_n, \dots, s_1) &= \sum_{m=1}^n \tilde{x}_{n+1,m}^{j,i}(y_{n+1}, s_{n,m}, sT_{m-1}) \\ \tilde{x}_{n+1,m}^{j,i}(y_{n+1}, s_{n,m}, sT_{m-1}) &\equiv FT_{n+1,m}^{j,i}(y_{n+1}, s_{n,m}) \tilde{u}_{m,i}(s_m \oplus sT_{m-1})\end{aligned}\quad [24]$$

If we define the functions Q' such that

$$FT_{n+1,m}^{j,i}(y_{n+1}, s_{n,m}) \equiv \frac{Q_{j \leftarrow i}^{n+1,m}(y_{n+1}, s_{n,m})}{P^m(s_m)} \quad [25]$$

it is shown in the next section that the Q' functions obey the following recurrence relations, that allow its computation:

$$Q_{j \leftarrow i}^{m+1,m}(y_{n+1}, s_{n,m}) = Q_{j \leftarrow i}^{m+1,m+1}(y_{n+1}, s_{n,m+1}) \sum_{k_{m+1}=0}^{N-1} \left[\frac{Q_{j \leftarrow i}^{m+1,m+1}(y_{n+1}, s_{n,m+2}, s_{m+1} \oplus k_{m+1})}{Q_{j \leftarrow i}^{m+1,m+1}(y_{n+1}, s_{n,m+1})} - \frac{P^{m+1}(s_{m+1} \oplus k_{m+1})}{P^{m+1}(s_{m+1})} \right] R_{k_{m+1}}^{m+1/2,i}(s_m) \quad [26]$$

for $m=1, \dots, n$, and

$$Q_{j \leftarrow i}^{m+1,n+1}(y_{n+1}) = Q_{j \leftarrow i}^{n+1}(y_{n+1}) \quad FT_{n+1,n+1}^{j,i}(y_{n+1}, s_{n,m}) = 1 \quad [27]$$

3.2. Proof of the recurrence relations.

The proof comes from induction reasoning; for the first two intervals, equations [25], [26] and [27] are consistent with equation [23]. On the other hand, if we consider equations [25], [26] and [27] true up to interval n , we have:

$$\tilde{x}_q^{n,i}(s_{n,1}) = \sum_{m=1}^n \frac{Q_{q \leftarrow i}^{m,m}(s_{n,m})}{P^m(s_m)} \tilde{u}_{m,i}(sT_m) \quad [28]$$

Replacing in r.h.s of equation [20]:

$$\tilde{x}_j^{n+1,i}(y_{n+1}, s_{n,1}) = G_{j \leftarrow i}^{n+1}(y_{n+1}) \tilde{u}_{n,i}(y_{n+1} \oplus sT_n) + \sum_{q=1}^N G_{j \leftarrow q}^{n+1}(y_{n+1}) \sum_{m=1}^n \frac{Q_{q \leftarrow i}^{m,m}(s_{n,m})}{P^m(s_m)} \tilde{u}_{m,i}(sT_m) \quad [29]$$

Comparing this expression with eq. [24] and [25], we can identify:

$$Q_{j \leftarrow i}^{m+1,m}(y_{n+1}, s_{n,m}) = \sum_{q=1}^N G_{j \leftarrow q}^{n+1}(y_{n+1}) Q_{q \leftarrow i}^{m,m}(s_{n,m}) \quad [30]$$

These recurrence relations of Q' are assumed true up to interval n , so:

$$\begin{aligned}Q_{j \leftarrow l}^{m+1,m}(y_{n+1}, s_{n,m}) &= \\ &= \sum_{q=1}^N G_{j \leftarrow q}^{n+1}(y_{n+1}) \left\{ Q_{q \leftarrow i}^{m,m+1}(y_{n+1}, s_{n,m+1}) \sum_{k_{m+1}=0}^{N-1} \left[\frac{Q_{q \leftarrow i}^{m,m+1}(y_{n+1}, s_{n,m+2}, s_{m+1} \oplus k_{m+1})}{Q_{q \leftarrow i}^{m,m+1}(y_{n+1}, s_{n,m+1})} - \frac{P^{m+1}(s_{m+1} \oplus k_{m+1})}{P^{m+1}(s_{m+1})} \right] R_{k_{m+1}}^{m+1/2,i}(s_m) \right\} = [31] \\ &= Q_{j \leftarrow i}^{m+1,m+1}(y_{n+1}, s_{n,m+1}) \sum_{k_{m+1}=0}^{N-1} \left[\frac{Q_{j \leftarrow i}^{m+1,m+1}(y_{n+1}, s_n, \dots, s_{m+1} \oplus k_{m+1})}{Q_{j \leftarrow i}^{m+1,m+1}(y_{n+1}, s_{n,m+1})} - \frac{P^{m+1}(s_{m+1} \oplus k_{m+1})}{P^{m+1}(s_{m+1})} \right] R_{k_{m+1}}^{m+1/2,i}(s_m)\end{aligned}$$

for $m=1, \dots, n-1$. Finally, for $m=n$, using property [A7]:

$$\begin{aligned}
 Q_{j \leftarrow i}^{m+1,n}(y_{n+1}, s_n) &= \sum_{q=1}^N G_{j \leftarrow q}^{n+1}(y_{n+1}) Q_{q \leftarrow i}^{m,n}(s_n) = \sum_{q=1}^N G_{j \leftarrow q}^{n+1}(y_{n+1}) Q_{q \leftarrow i}^n(s_n) = \\
 &= \sum_{q=1}^N G_{j \leftarrow q}^{n+1}(y_{n+1}) \sum_{k=0}^{N-1} Q_{qk}^{n+1,i} R_k^{n+1/2,i}(s_n) = \\
 &= Q_{j \leftarrow i}^{n+1}(y_{n+1}) \left[\frac{Q_{j \leftarrow i}^{n+1}(y_{n+1} \oplus k)}{Q_{j \leftarrow i}^{n+1}(y_{n+1})} - \frac{P^{n+1}(y_{n+1} \oplus k)}{P^{n+1}(y_{n+1})} \right] R_k^{n+1/2,i}(s_n)
 \end{aligned} \tag{32}$$

3.3. Consistency checks.

As a significant application, consider the case when $Q_{j \leftarrow i}^n(s) = Q_{j \leftarrow i}(s)$ are independent on the interval.

Then, because there is no discontinuity

$$R_{k_{m+1}}^{m+1/2,i}(s_m) = s_m^{k_{m+1}} \tag{33}$$

and

$$\begin{aligned}
 \frac{Q_{j \leftarrow i}^{m+1,m}(y_{n+1}, s_{n,m})}{P^m(s_m)} &= \frac{Q_{j \leftarrow i}^{m+1,m+1}(y_{n+1}, s_{n,m+1})}{P^m(s_m)} \left[\frac{Q_{j \leftarrow i}^{m+1,m+1}(y_{n+1}, s_{n,m+2}, s_{m+1} \oplus s_m)}{Q_{j \leftarrow i}^{m+1,m+1}(y_{n+1}, s_n, \dots, s_{m+1})} - \frac{P^{m+1}(s_{m+1} \oplus s_m)}{P^{m+1}(s_{m+1})} \right] = \\
 &= \frac{P^{m+1}(s_m)}{P^m(s_m)} \left[\frac{Q_{j \leftarrow i}^{m+1,m+1}(y_{n+1}, s_{n,m+2}, s)}{P^{m+1}(s)} \right]_{s=s_{m+1} \oplus s_m}
 \end{aligned} \tag{34}$$

This expression shows that:

a. Consistency with reference [1], section 4.3.

If $Q_{j \leftarrow i}(s) = 1$ we get back the results of reference [1] section 4.3 as may be easily checked.

b. Consistency with a single interval partitioned in several subintervals.

If also the polynomials $P^m(s)$ are independent of the interval, i.e., $P^m(s) = P(s)$ we get

$$\frac{Q_{j \leftarrow i}^{m+1,m}(y_{n+1}, s_{n,m})}{P(s_m)} = \left[\frac{Q_{j \leftarrow i}^{m+1,n+1}(s)}{P(s)} \right]_{s=y_{n+1} \oplus sT_n} = \left[\frac{Q_{j \leftarrow i}(s)}{P(s)} \right]_{s=y_{n+1} \oplus sT_n} = \left[G_{j \leftarrow i}(s) \right]_{s=y_{n+1} \oplus sT_n} \tag{35}$$

which leads to the correct result when computing \bar{x} , i.e.

$$\tilde{x}_j^{n+1,l}(s_{n+1}, s_{n,1}) = \sum_{m=1}^{n+1} G_{j \leftarrow i}(sT_{n+1}) \tilde{u}_m(sT_m) = \left[G_{j \leftarrow i}(s) \tilde{u}_m(s) \right]_{s=y_{n+1} \oplus sT_n} \tag{36}$$

just equivalent to the one interval transfer function result when expressed in terms of partitions.

IV. SUMMARY OF RESULTS.

In order to give a complete approach to the problem exposed in the introduction of section 2, we must consider the general case where $[B_{n+1}] \neq [I_N]$. The recurrence relations of equation [26] are still valid as long as we modify two of the ingredients involved in their computation:

$$\begin{cases} [A_n] \rightarrow [A_n^{eq}] = [B_n]^{-1} [A_n] [B_n] \\ [R^{n+1/2,i}] \rightarrow [R_{eq}^{n+1/2,i}] = [Q^{n+1,i}]^{-1} [B_{n+1}]^{-1} [B_n] [Q^{n,i}] \end{cases} \tag{37}$$

The first equation of [37] takes into account that we dropped the “eq” superscript of the matrix $[A_n^{eq}]$ in equation [3], and the second one is a generalization of equation [16] when we consider $[B_n] \neq [I_N]$ (see the general boundary conditions equation [4]).

Figure 2 summarizes the TFT approach, indicating that transmission functions are products of transfer function matrices in different Laplace variables. FTs are an extension of linear systems to piecewise linear ones that keeps the nice features for the treatment of model topology division within each time interval. The diagram in figure 3 summarizes the process to compute the output variables via the FT recurrence relations approach, as an alternate to the FT matrix approach, with only involving the input–output pairs in the transfer function matrices associated to input i and output j. In this form the theory is able to find very fast algorithms for their computation in the time domain.

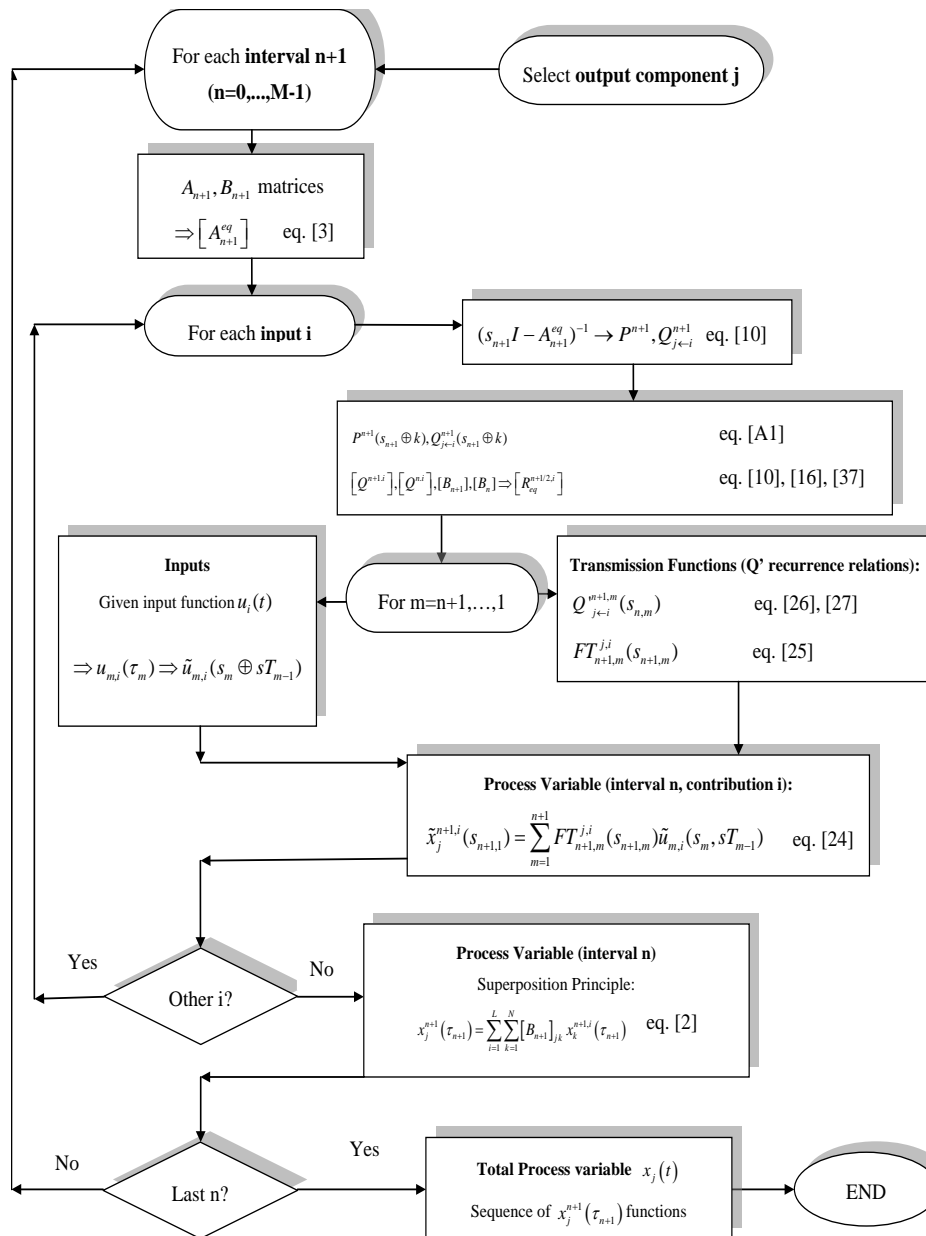


Figure 3. Process to compute the output via FT recurrence relations approach

V. VERIFICATION THROUGH EXAMPLES.

In this section several examples of piecewise linear MIMO systems are considered and their corresponding process variables are calculated following several methods (including the process described in the diagram of figure3) in order to mutually compare their results, thus ensuring the validity of the transmission functions techniques.

5.1. Analytical example.

The following simple example is chosen to clarify the main results. Consider the piecewise linear system defined in $\mathbb{R}_{2 \times 2}$ by

$$\dot{\bar{x}}(\tau_n) = \left[\lambda^n \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} - \begin{pmatrix} 0 & \beta^n \\ 0 & 0 \end{pmatrix} \right] \bar{x}(\tau_n) + \begin{pmatrix} u_n(\tau_n) \\ 0 \end{pmatrix} \quad 0 < \tau_n \leq \Delta T_n \quad n = 1, 2, 3, 4 \quad [38]$$

with C a fixed matrix and β^n, λ^n piece-dependent parameters. Boundary conditions between pieces are such that x is continuous in time and we assume zero initial conditions.

The characteristic polynomials, the Q polynomials and the discontinuity vector of the interval n can be obtained by [10] and [15] respectively.

As an example, the analytical expression of component 1 in the fourth interval is shown below, derived for the case where $C_{11} = -1, C_{12} = 1, C_{21} = 1, C_{22} = -1, \lambda^n = 1/n, \beta^n = 3$, and the constant input signals $u_1=u_3=1, u_2=u_4=0$:

$$\bar{x}_1^4(s_4, s_3, s_2, s_1) = \frac{[-5/6] + [1/4]s_3 + [1/3]s_4 + s_3s_4}{P_4(s_4)P_3(s_3)s_2s_1} + \left\{ \begin{array}{l} \frac{[1/12] + [-29/12]s_2 + [-13/4]s_3 + [-10/3]s_4 + [-5/2]s_2s_3 + [-7/3]s_2s_4 + [-2]s_3s_4 + s_2s_3s_4}{P_4(s_4)P_3(s_3)P_2(s_2)s_1P_1(s_1)} \\ + \frac{[-29/24] + [-5/6]s_2 + [-5/4]s_3 + [-7/6]s_4 + [1/4]s_2s_3 + [1/3]s_2s_4 + [1/2]s_3s_4 + s_2s_3s_4}{P_4(s_4)P_3(s_3)P_2(s_2)P_1(s_1)} \end{array} \right\} \quad [39]$$

For details of the calculation up to the 3rd interval see appendix A.4.

5.2. Computer-aided testing of the approach.

To check the validity of the theory, we developed a number of computer routines, built in MATLAB, that compute transient piecewise MIMO systems in a variety of dynamics. They implement the following alternate methods, applied to the examples described below.

1. FT matrix method. This method implements the FT matrix approach solution of equation [12] that is feasible for these low order few intervals case. This is considered the reference method to compare with.
2. Frobenius method. The MATLAB FT matrix method above may also be applied to the \bar{z} canonical variables, as follows. Equation [15] shows that, because vector \bar{z} corresponds to a canonical system, the results within interval n of vector $\bar{z}^{n,i}(T_n)$, corresponds to the canonical case described by [16]. On the other hand, eq. [16] can be written as

$$z_k^{n+1}(T_n) = \sum_{m=0}^{N-1} \left[R^{n+1/2} \right]_{k,m} z_m^n(T_{n-1} + \Delta T_n) \quad [40]$$

and it shows that if the MATLAB results at T_n of iteration n in the FT matrix method are multiplied by matrix R, they provide the right initial conditions for the next interval, and it should occur that

$$x_j^{n+1}(T_n + \tau) = \sum_{m=0}^{N-1} \left[Q^{n+1,l} \right]_{j,m} z_m^{n+1}(T_n + \tau) \quad [41]$$

The left hand side is evaluated according to [12] (FT matrix method), while the right hand side use the same FT matrix method, but applied to the z variables as described above.

3. FT recurrence relations method. It solves first the recurrence relations of the transmission functions equations [25], [26] and [27], then compute outputs for given inputs as a separate process. For the analytical example in particular, the recurrence relations have been solved by hand and the MATLAB routine evaluates the resulting expressions (for instance, equation [39]).

5.2.1. Analytical example.

The following values for the parameters have been chosen:

$$C_{11} = -1, C_{12} = 1, C_{21} = 1, C_{22} = -1, \lambda_1 = 1, \lambda_2 = 1/2, \lambda_3 = 1/3, \lambda_4 = 1/4, \beta_n = 3.$$

We set a double step function as input signal. The limits of the intervals are set at: $T_1 = 2s, T_2 = 7s, T_3 = 17s$. In this simple case it is viable to calculate the output components expressions by hand. The graphical comparisons of the results between the methods are shown below (figures 4 and 5). The error is extremely low (absolute value of the difference goes from 10^{-14} to 10^{-13} , around 10^{-12} % of the output signal values).

5.2.2. Point Kinetics Nuclear Reactor model.

To address a more realistic case we consider now a sequence of nuclear chain reaction transitions due to control rod insertions-extractions in startup tests of nuclear facilities. This example is chosen because there are many studies on the Nuclear Reactor Point Kinetics (PK) model (see reference [8]) The PK without feedback is an example of MIMO piecewise linear systems as follows.

The vector of process variables includes a set of concentrations of a family of radioactive isotopes generated as a result of fissions products, able to produce delayed neutrons, together with the very fast additional neutron production that is direct result of the fissions themselves. Fissions are generated with a multiplicative factor, the reactivity $\rho(t)$, from a neutron flux $\phi(t)$, basically proportional to the total number of fissions. Then, the corresponding state vector is described by components:

$$\begin{cases} x_j(t) = C_j(t) & \text{Precursor isotope concentration of family } j \text{ (} j=1, \dots, N \text{)} \\ x_N(t) = \phi(t) & \text{Nuclear reactor neutron flux} \end{cases} \quad [42]$$

The features of the multiplicative reactor change $\rho(t)$. In order to measure ρ , a set of experiments changing its value $\rho(t) = \rho_n$ during given time intervals $T_{n-1} < t < T_n$, are performed, inferring the reactivity values from the time evolution of the neutron flux. The experiments are made at sufficiently low flux level as to prevent that any feedback mechanism introduces simultaneous, intrinsic reactivity changes. The experiment thus fails if the neutron flux reaches an excessive value.

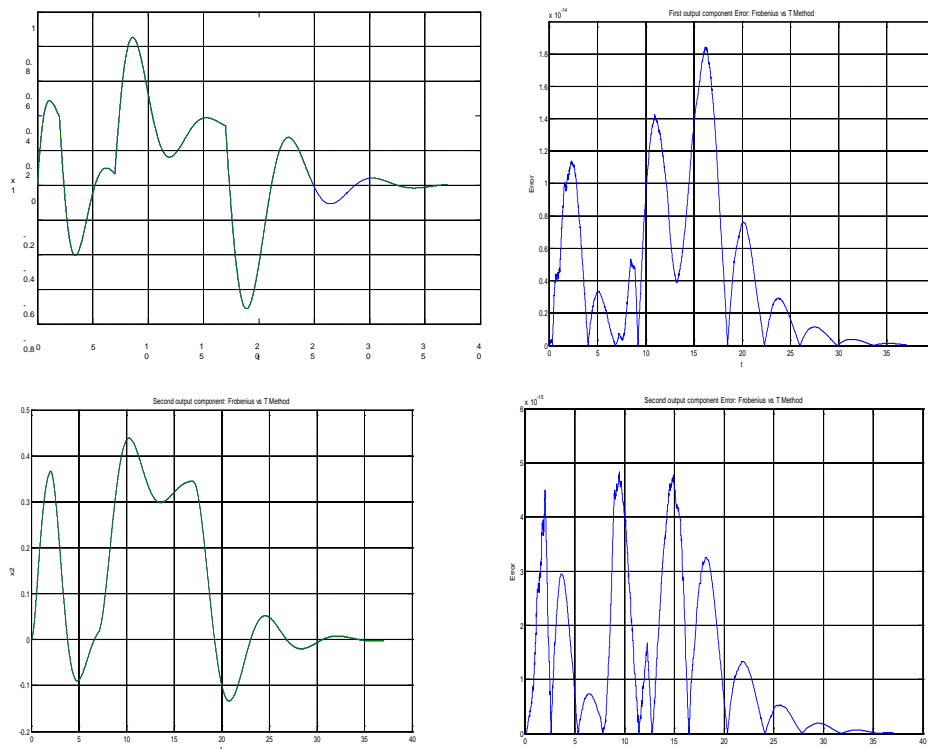


Figure 4: Analytical example– Frobenius and FT matrix method comparison (x vs time and absolute error): (up) First output component; (down) Second output component.

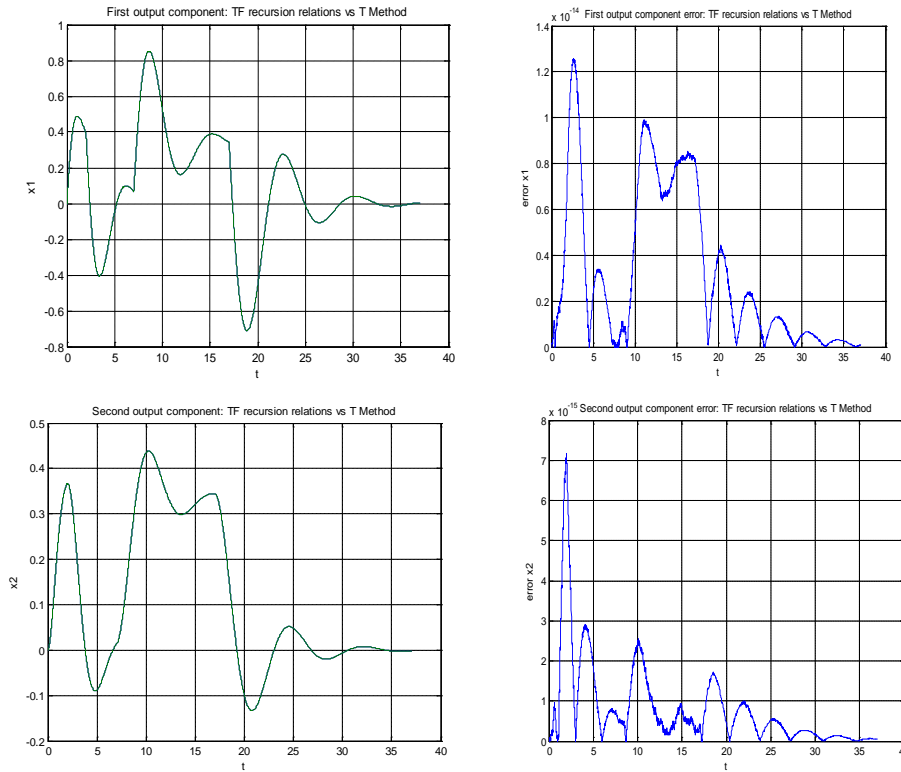


Figure 5: Analytical example – FT recurrence relations and FT matrix method comparison (x and absolute error vs time) (up) First output component; (down) Second output component.

5.2.2.1. Six Groups Model.

We can consider a six precursor groups PK model in a three intervals sequence⁶:

- First interval has null initial conditions, is subcritical, and a constant source input signal is applied.
- Second interval lasts 400 time units, is supercritical and input signal is maintained.
- Third interval lasts 600 time units, is subcritical and the input signal is still maintained.

Taking into account the previous hypothesis, the dynamical matrices of the system are:

$$[A_n] = \begin{pmatrix} \beta_1 / \ell & -\lambda_1 & 0 & 0 & 0 & 0 & 0 \\ \beta_2 / \ell & 0 & -\lambda_2 & 0 & 0 & 0 & 0 \\ \beta_3 / \ell & 0 & 0 & -\lambda_3 & 0 & 0 & 0 \\ \beta_4 / \ell & 0 & 0 & 0 & -\lambda_4 & 0 & 0 \\ \beta_5 / \ell & 0 & 0 & 0 & 0 & -\lambda_5 & 0 \\ \beta_6 / \ell & 0 & 0 & 0 & 0 & 0 & -\lambda_6 \\ (\rho^n - 1) \frac{\beta}{\ell} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \end{pmatrix} \quad [43]$$

$$[B_n] = [I_7]$$

and where

$$\begin{cases} 1 / \lambda_j = \text{average time of the group } j \text{ to produce a delayed neutron after any fission} \\ \beta_j = \text{fraction of the total number of neutrons per fission born from decay of group } j \\ \beta = \sum_{j=1}^6 \beta_j = \text{total delayed fraction} \\ \ell = \text{period time for fast neutrons to born after a fission} \\ \rho^n = \text{reactivity of interval } n \text{ in dollars } (\$ = \beta) \end{cases} \quad [44]$$

⁶ Values of parameters, times, and input functions are in range but arbitrary, just for demonstration purposes. They are not representative of any actual reactor or transient.

The routine developed implements the application of Frobenius method, solves the FT recurrence relations and generates the graphical comparisons with the FT matrix method results. A realistic set of parameters (to a certain extent) has been chosen, and a change of the total fraction of delayed neutrons β from 0.007 to 0.006 in the last interval has been simulated to introduce a deeper discontinuity.

e	β_1	β_2	β_3	β_4	β_5	β_6	ℓ	$\rho(\$)$
1	0,0003	0,0013	0,0011	0,0024	0,0012	0,0006	$4 \cdot 10^{-6}s$	-0.01
2	0,0003	0,0013	0,0011	0,0024	0,0012	0,0006	$4 \cdot 10^{-6}s$	+0.01
3	0,0002	0,0012	0,0010	0,0021	0,0010	0,0005	$4 \cdot 10^{-6}s$	-0.02

and $\lambda_1 = 0,0133 \text{ s}^{-1}$; $\lambda_2=0,0325 \text{ s}^{-1}$; $\lambda_3=0,1219 \text{ s}^{-1}$; $\lambda_4=0,3169 \text{ s}^{-1}$; $\lambda_5=0,9886 \text{ s}^{-1}$; $\lambda_6=2,9544\text{s}^{-1}$.

As expected, the error is way higher than before, since the amount of operations has increased significantly. In any case its value is around $10^{-8} \%$.

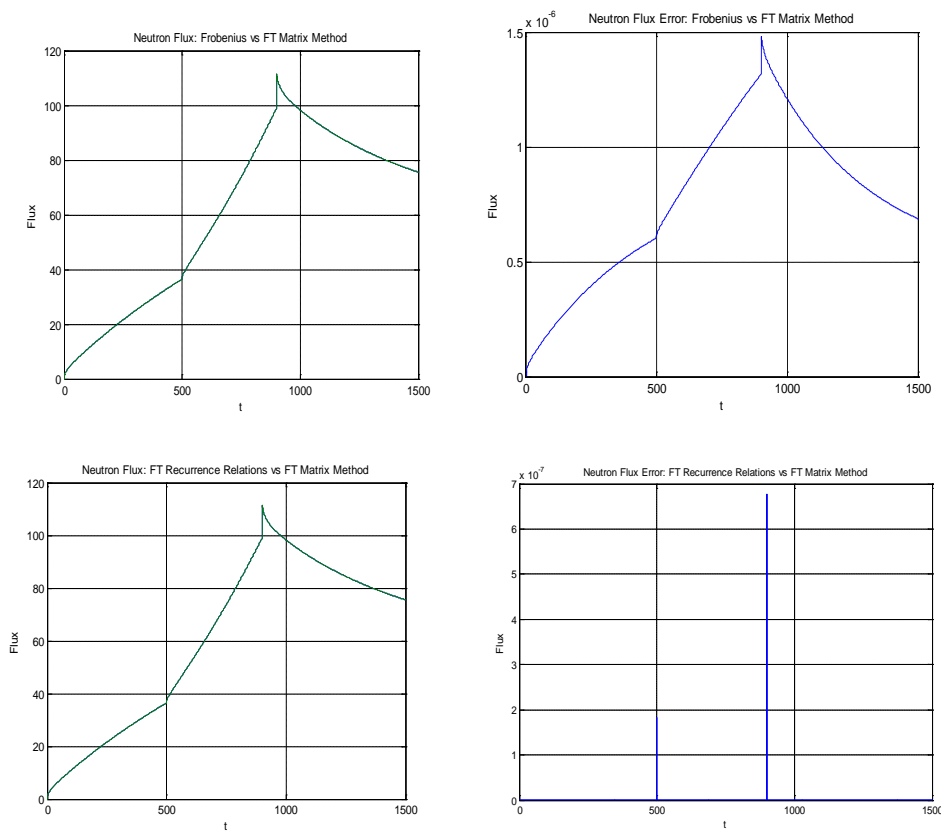


Figure 6: Point Kinetics 6 groups/3 intervals: Methods comparison (flux and absolute error vs time) (up) Frobenius vs FT matrix method; (down) FT recurrence relations vs FT matrix method.

This no feedback PK model illustrates well the capability of the Transmission Functions method to reproduce discontinuities. Figure 6 shows the results when different sojourn times in the three intervals are taken and the input is the same step in the three intervals. Because the $[R^{1+1/2}]$ matrix of the two first intervals is unity (the $Q_{j \leftarrow i}^n(s_n)$ polynomials may easily be checked to be the same for $n=1, 2$, as well as the input) no discontinuity in derivatives is expected, while discontinuous derivatives should appear in-between the second and third intervals due to different beta fractions, inducing $[R^{2+1/2}]$ different than unity. However, as seen in the figure, apparent steps (discontinuity in the variables) at the transitions times are also observed. Those are actually due to the stiffness (large differences in the poles of the polynomials $P^n(s_n)$ in the denominators of the transfer functions) of the PK model that includes large negative poles.

In summary we have three different sources of discontinuities, clearly discriminated by the theory:

- i. Discontinuities of the input.

- ii. Apparent discontinuities in the variables due to stiffness of $P^n(s_n)$.
- iii. Discontinuities of the variable derivatives due to $[R^{n+1/2}]$ matrix differing from identity.

VI. DISCUSSION AND CONCLUSIONS

6.1. System dynamics remarks.

The main purpose of these developments is not to research new findings in complex system dynamics, rather to exploit them within the context of Protection Engineering problems. Nonetheless, some system dynamics remarks are worth the point:

- i. The FT recurrence relations method provides, for piecewise linear systems, a projection into a single variable that represents the prior history of the transients through the FT. That is, it is close to the integral, rather than differential methods. It is immediately suggested the connexion, as limiting cases of infinite number of infinitely small intervals, with the tradition of the Wiener-Volterra (see references [9] and [10]) approaches to nonlinear systems.
- ii. Sudden changes in trends, i.e. in derivatives of process variables, are the main observables to indicate the occurrence of dynamic transitions, very often a consequence of protective actions and/or of the onset of degrading phenomena. This justifies the study of sequences of events, that are a characteristic feature of Protection Engineering problems. Continuous behaviour between events is expected, very often the results of control system actions, that, as shown in this paper, may be interpreted as the infinite limits mentioned in point (i) above, when the continuity condition $[R] = [I_N]$ is applied to each interval.
- iii. Among the uncertainties in boundary conditions, fluctuating inputs due to noise (or high statistics) are often involved. Because of the particularly attractive features of the response of piecewise linear systems to those stochastic inputs, (see reference [11]) the extension of these inputs to transmission functions opens new possibilities.

6.2. Risk assessment. PSA implications.

Although many other applications may be envisaged, the main motivation for this research has been the development of a methodology and tools to assess the quality of risk assessments made with the well known, PSA (Probabilistic Safety Assessment) method, widely used in several industries as an approach to risk assessment (see references [12], [13], [14], [15], [16], [17], [18], [19]). It has become customary in certain domains, like in Nuclear Engineering, as part of the regulatory process to verify compliance of Protection Designs. This widespread method substantially consists of three well defined stages, namely

1. Delineate the possible sequences, which amount to find all SOTs of potential relevance.
2. Determine sequence success criteria, such that the sequence of configurations of the safety systems responsible to successfully change with their intervention the wrong trends of the damage indicators evolution in the different sequences, are identified and discriminated versus other configurations not able to do it (failed configurations).
3. Compute the sequence failure frequency of its safety systems configurations, by using, for instance, FT/ET techniques.

While detailed methods and abundant literature (see [14], [18]) provides guidance for stage 3, such a guidance become loose when describing stages 1 and 2, mainly due to the unique phenomena involved in each application domain, their strong nonlinearities, and their dependence on the protection design methods, usually very sophisticated and technology dependent (as for instance use of Design Basis Transients, [20]).

The direct motivation for this paper was to be able to understand stages 2 and 3 in a piecewise-linear PSA, i.e., one where the dynamics involved in the transitions are piecewise linear. The lessons learned may then be translated into the general case.

A major problem with this industry approach is that in order to analyze consequences (i.e., to characterize success) with accident computer codes, it is necessary to work with single, deterministic transients. And in order to cover all potential situations and to compute their frequencies, probabilistic techniques are necessary, that is, to consider groups of transients (specifically, all those covering the uncertainties) and to apply statistical methods. For instance, in reference [17] the drawbacks of the transient modelling in the reliability treatment of electric systems (cascades effects) are clearly stated.

MIMO FTs may contribute to a better approach to this problem, via the optimization of the process to identify sequence failure domains, i.e., the set of SOTs that exceed safety limits.

In fact, it allows computing the subset of output stimuli variables, subject to constraints in the inputs, in a myriad of transients, by using first a numerical technique that substantially pre-computes and store the transmission functions independently on inputs. The FTs are accounting for the influence of the rest of the

system variables in that particular variable set, without requiring information other than the one stored in the FTs that may be pre-processed.

The FT approach is also of interest to help the process to verify adequate success criteria in the current Probabilistic Safety Assessment (PSA) methodology (see [14]). That is, to identify minimum inputs and its timing in those variables associated to the system safety functions (of any protective action, either automatic or manual) that are required for success. For instance, if the inputs are actually boundary conditions that isolate the plant model from the outside represented by a set of safety systems, so that the boundary condition variables may be interpreted as safety functions of safety systems, the approach may be used to identify minimum requirements for success for each sequence system function variables (i.e., minimum flows, temperature and pressure ranges, etc.).

They also facilitate the application of the TSD theory to the calculation (see [3]) of the contribution to the frequency of any SOT within the failure domain in order to weight them with the FT/ET results of stage 3 and to aggregate them to find the frequency of exceeding safety limits.

Then, it is enough to apply an appropriate numerical scheme to invert into the time domain the application of the FT operators to each of the time input shapes, keeping in memory the FT numerical representation.

6.3. Conclusions. Future research.

The theory of transmission functions is able to implement useful techniques to model the complex aspects involved in the process to identify failure domains and to verify success criteria in a PSA process. They extend the many useful properties of classical transfer functions for linear systems (that are but the case of a single interval) allowing the propagation of the dynamic effects from one interval to another (see figure 2).

This paper is part of a research project trying some new risk assessment techniques in the context of protection engineering (see [1], [3], [5]). Altogether, they aim to be able to quantitatively validate and verify the adequacy of designed protections, providing a generalization and mathematical interpretation of all stages of the PSA process in a way independent on the domain of application, with particular emphasis in the envelope issue. This paper has dealt with the uncertainty in boundary conditions and time of protective actions, as well as with the validation of success criteria, and the techniques described are able to provide the sequence damage and failure domains associated to them.

Next step will clarify more the relationship with the PSA sequences and will provide a numerical method to convert these results into useful algorithms (see [21]). At the same time, it will be shown the extension of the algorithms to cases where the matrices defining the piece dynamics are of infinite dimension, so that transfer functions are no longer rational fractions, a situation often appearing when the original piecewise linear models come from partial differential equations, rather than ordinary ones. Future steps will be focused on the identification (see [22]) of envelope transmission functions starting from an adequate, representative, best estimate, transient data base and to the computation of the exceedance frequency.

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Glossary and Notations

CSISO, Canonical Single-Input Single-Output, Phase variable/Frobenius

MIMO, Multi-Input Multi-Output

PK, Nuclear Reactor Point Kinetics model

PSA, Probabilistic Safety Assessment

SOT, Sequence of Transitions

TSD, Theory of Stimulated Dynamics

FT/ET, Fault Tree/Event Tree

FT, Transmission Functions

$[A]$, matrix A

$[A]_{i,j}$, element corresponding to the i row, j column of the matrix A

$[A][B] \equiv \sum_{k=1}^N [A]_{ik} [B]_{kj}$, usual product of matrices A and B

$x_j(t)$, time evolution of the component j of the N-th order process vector \bar{x}

T_n , n-th event time when a transition in the dynamics has been induced

$\Delta T_n \equiv T_n - T_{n-1}$, duration of the n-th piece of a piecewise linear system

$[A_n], [B_n]$, state description matrices of the n-th piece-wise linear dynamics of sequence

$[I_N]$, identity matrix of dimension N x N

$\delta_{j,i}$, Kronecker delta

$u_{n,i}(t)$, i-th input signal of the system during the n-th piece of a piece-wise linear system.

$G_{j \leftarrow i}^n(s_n)$, transfer description of $x_j(t) - \hat{u}_{n,i}(t)$ relationship for the n-th piece-wise linear dynamics

$P^n(s), Q_{j \leftarrow i}^n(s)$, denominator and numerator of $G_{j \leftarrow i}^n(s_n)$

$\tilde{\varphi}(s_1, s_2, \dots)$, multiple Laplace transform of function $\varphi(t_1, t_2, \dots)$ of the independent variables t_1, t_2, \dots , where s_n is the Laplace variable associated to the partition time t_n

\oplus , partition operator (defined in equation [6])

$Q_{j,k}^{n,i}$, piecewise linear Frobenius transformation (defined in [15]) that reduces the system in variables \bar{x} to a piecewise CSISO Frobenius canonical form in variables \bar{z}

$FT_{n,m}^{j,i}(sT_{n,m})$, Transmission Functions (FT) of the process variable x_j due to input i (see equation [12])

$sT_{n,m}$, Laplace variable associated to the time variable $T_{n,m} \equiv \sum_{j=m}^n t_j$

$s_{n,m} \equiv (s_n, \dots, s_m)$, collection of local Laplace variables corresponding to the m -th to n -th pieces of a piecewise linear system

τ_n , local time variable of the n -th piece of a piecewise linear system

y_{n+1} , local Laplace variable associated to the local time variable τ_n

$\tilde{u}_{n+1,i}(y_{n+1} \oplus sT_n)$, Laplace transform of the $u_{n+1,i}(\tau_{n+1} + T_n)$ time function

$P^n(s \oplus k)$, truncated polynomial vector, derived from polynomial P

$R_k^{n+1/2,i}(s), R_{kl}^{n+1/2,i}$, discontinuity vector and matrix between intervals n and $n+1$

$p_{n,k} Q_{jk}^{n,i}$, coefficients of the $P^n(s), Q_{j \leftarrow i}^n(s)$ polynomials

$z_k^{n+1,i}$, k component of the canonical variables vector of the n -th piece of a piecewise linear system

$[Q \otimes P]$, cross polynomial (see section A.1.2 of appendix)

$L[\varphi(t)]$, Laplace transform of $\varphi(t)$ function

Appendix A: Transfer functions in partitions of time intervals

In this appendix we provide concepts and properties of transfer functions. In A.1 we show relations that are derived from polynomials and properties they satisfy, and in A.2 properties of cross polynomials that are responsible for coupling intervals

A.1. Notation and properties of polynomials in partitions.

A.1.1. Properties of truncated polynomials $P(s \oplus k), P(k \oplus s), P_k^T(s), \bar{P}_k^T(s)$.

For any polynomial $P(s) \equiv s^N + \sum_{k=0}^{N-1} p_k s^k$, of order N , we will use the notations

$$P(s_1 \oplus s_2) \equiv \frac{P(s_1) - P(s_2)}{(s_2 - s_1)} \equiv \sum_{k=0}^{N-1} P(s_1 \oplus k) s_2^k \equiv \sum_{k=0}^{N-1} P(k \oplus s_2) s_1^k$$

$$\sum_k P(s_1 \oplus k) s_2^k = \frac{P(s_1) - P(s_2)}{s_2 - s_1} = \sum_{j=1}^N p_j \frac{s_1^j - s_2^j}{s_2 - s_1} = - \sum_{j=1}^N p_j \sum_{m=0}^{j-1} s_2^m s_1^{j-m-1} = \sum_{k=0}^{N-1} \left(- \sum_{m=k+1}^N p_m s_1^{m-k-1} \right) s_2^k$$

$$P(s_1 \oplus k) = - \sum_{m=k+1}^N p_m s_1^{m-k-1} = - \sum_{m=0}^{N-1-k} p_{k+m+1} s_1^m \equiv -s_1^{-(k+1)} \bar{P}_k^T(s_1)$$

[A1]

$$\begin{cases} P_k^T(s) \equiv \sum_{m=0}^k p_m s^m \\ P(s \oplus k) \equiv P(k \oplus s) = -s^{-(k+1)} \bar{P}_k^T(s) \\ \bar{P}_k^T(s) = \sum_{m=k+1}^N p_m s^m = P(s) - P_k^T(s) \end{cases}$$

A.1.2. Properties of cross polynomials $[Q \otimes P](s \oplus k), [P \otimes Q](s \oplus k)$

Also we define, with Q a polynomial of order $N-1$ (i.e., $q_N = 0, p_N = 1$)

$$\begin{aligned} G(s) &= \frac{Q(s)}{P(s)} \\ PTF(s_1 \oplus s_2) &\equiv \frac{P(s_1 \oplus s_2)}{P(s_1)} \\ PTF(s \oplus k) &\equiv \frac{P(s \oplus k)}{P(s)} \end{aligned} \tag{A2}$$

$$\begin{aligned} \mathbb{Z}(s \oplus k) &= QTF(s \oplus k) - PTF(s \oplus k) = \frac{[Q \otimes P](s \oplus k)}{Q(s)P(s)} \\ [Q \otimes P](s \oplus k) &\equiv -[P \otimes Q](s \oplus k) = [Q(s \oplus k)P(s) - P(s \oplus k)Q(s)] \end{aligned}$$

We note the following conclusion

$$\begin{aligned} [Q \otimes P](s \oplus k) &= Q(s \oplus k)P(s) - P(s \oplus k)Q(s) = \\ &= Q(s \oplus k)P_k^T(s) - P(s \oplus k)Q_k^T(s) = \\ &= s^{-(k+1)} [Q_k^T(s)\bar{P}_k^T(s) - \bar{Q}_k^T(s)P_k^T(s)] \\ &= \sum_{m=0}^k \sum_{l=k+1}^N (q_l p_m - p_l q_m) s^{l+m-k-1} \quad (q_N = 0, p_N = 1) \\ \left\{ \begin{aligned} [Q \otimes P](s \oplus k) &= \sum_{n=0}^{N-1} \langle qp \rangle_n^k s^n \\ \langle qp \rangle_n^k &= \sum_{l=\max\{k+1, n+1\}}^{\min\{n+k+1, N\}} (p_l q_{n+k+1-l} - q_l p_{n+k+1-l}) = \langle qp \rangle_k^n \end{aligned} \right. \end{aligned} \tag{A3}$$

Since the previous coefficient matrix is symmetric, we only need to compute $\frac{(N+1)N}{2}$ out of the

$N \times N$ elements, and the rest are fixed by symmetry.

A.2. Transfer functions and cross polynomials.

With the definition and relations

$$\begin{aligned} G_{j \leftarrow i}(s \oplus z) &\equiv \frac{G_{j \leftarrow i}(s) - G_{j \leftarrow i}(z)}{z - s} = \left[\frac{[s[I] - [A]]^{-1} - [z[I] - [A]]^{-1}}{z - s} \right]_{ji} \\ &= \sum_{q=1}^N [s[I] - [A]]_{jq}^{-1} [z[I] - [A]]_{qi}^{-1} = \sum_{q=1}^N G_{j \leftarrow q}(s) G_{q \leftarrow i}(z) \end{aligned} \tag{A4}$$

the following properties of the transfer functions are proved

$$\begin{aligned} G_{j \leftarrow i}(s \oplus z) &= \sum_{m=1}^N G_{j \leftarrow m}(s) G_{m \leftarrow i}(z) = \left[\frac{Q_{j \leftarrow i}(w)}{P(w)} \right]_{w=s \oplus z} = \sum_{l=1}^N G_{j \leftarrow l}(s) \sum_{k=0}^{N-1} \mathbb{Z}_{l \leftarrow i}(s \oplus k) \frac{z^k}{P(z)} \\ \mathbb{Z}_{j \leftarrow i}(s \oplus k) &\equiv \frac{Q_{j \leftarrow i}(s \oplus k)}{Q_{j \leftarrow i}(s)} - \frac{P(s \oplus k)}{P(s)} \end{aligned} \tag{A5}$$

We also define $G_{j \leftarrow i}(s \oplus k)$ such that

$$\sum_k G_{j \leftarrow i}(s \oplus k) \frac{z^k}{P(z)} \equiv G_{j \leftarrow i}(s \oplus z) \tag{A6}$$

$$G(s \oplus k) = \sum_k \frac{[Q \otimes P]_{j \leftarrow i}(s \oplus k)}{P(s)}$$

$$\sum_{q=1}^N G_{j \leftarrow q}(s) Q_{qk}^i = G(s \oplus k) = Q_{j \leftarrow i}(s) Z_{j \leftarrow i}(s \oplus k) = \frac{[Q \otimes P](s \oplus k)}{P(s)} \tag{A7}$$

where Q_{qk}^i is defined in equation [10].

A.3. Some examples on the use of partition operators.

In this section we provide some explicit expressions of application of partition operators in the significant classical pole decomposition of rational transfer functions:

$$\begin{aligned} \tilde{\varphi}_1(s) &= Q(s) = \sum_{j=0}^{N-1} q_j s^j \\ \tilde{\varphi}_2(s) &= \frac{1}{P(s)} = \frac{1}{\sum_{j=0}^N p_j s^j} \end{aligned} \tag{A8}$$

In this case the rational transfer function can be expressed as:

$$\varphi_1(s)\varphi_2(s) = \frac{Q(s)}{P(s)} = \sum_{p=1}^N \frac{\sum_{j=0}^{N-1} q_j a_p^j}{s + z_p} \tag{A9}$$

where

$$\begin{aligned} \frac{s^j}{P(s)} &\equiv \sum_{p=1}^N \frac{a_p^j}{s + z_p} \\ a_p^j &= \frac{(-z_p)^j}{P'(-z_p)} \end{aligned} \tag{A10}$$

The application of partition operators to the case of two intervals, yields

$$[\varphi_1(s)\varphi_2(s)]_{s_1 \oplus s_2} = \sum_{p=1}^N \sum_{j=0}^{N-1} q_j \frac{\frac{a_p^j}{s_1 + r_p} - \frac{a_p^j}{s_2 + r_p}}{(s_2 - s_1)} = \sum_{p=1}^N \frac{\sum_{j=0}^{N-1} q_j a_p^j}{(s_1 + r_p)(s_2 + r_p)} \tag{A11}$$

A general expression that extends to n intervals is obtained by iterating the procedure above:

$$[\varphi_1(s)\varphi_2(s)]_{sT_{n,m}} = \sum_{p=1}^N \frac{\varphi_1(-z_p)}{\prod_{\substack{q=1 \\ q \neq p}}^N (z_q - z_p)} \frac{1}{\prod_{l=m}^n (s_l + z_p)} \tag{A12}$$

Here the numerator and denominator are polynomials in all s variables with the numerator degrees lower than those of the denominator, so they are regular rational fractions.

A.4. Use of the recurrence relations in the analytical example.

Let's consider system [38] in three intervals, choosing

$C_{11} = -1, C_{12} = 1, C_{21} = 1, C_{22} = -1, \lambda_n = 1/n, \beta_n = 3, u_{1,1}(\tau_1) = u_{3,1}(\tau_3) = 1, u_{2,1}(\tau_2) = 0$:

The system is defined then by:

$$[B_n] = [I_2], \quad [A_n] = \begin{pmatrix} -1/n & 1/n - 3 \\ 1/n & -1/n \end{pmatrix} \rightarrow [A_n^{eq}] = [B_n]^{-1} [A_n] [B_n] = [A_n] \tag{A13}$$

To apply the recurrence relations equation [26], we will need to compute some mathematical objects that might be unfamiliar for the reader. Hence, it is useful to show some explicit examples of their calculation for the system defined by [A13]. Let's fix input component $i=1$:

- Transfer functions and Q matrices:

$$(s_n I - A_n^{eq})^{-1} \rightarrow \begin{cases} P^n(s) = s^2 + 2s/n + 3/n \\ Q_{j \leftarrow i}^n(s) = \begin{pmatrix} 1/n + s & 1/n - 3 \\ 1/n & 1/n + s \end{pmatrix} \rightarrow Q_{j \leftarrow 1}^n(s) = \begin{pmatrix} 1/n + s \\ 1/n \end{pmatrix} \end{cases}$$

$$Q_{j \leftarrow 1}^n(s) = \sum_{k=0}^{N-1} Q_{jk}^{n,1} s^k = \begin{pmatrix} 1/n & 1 \\ 1/n & 0 \end{pmatrix} \begin{pmatrix} s^0 \\ s^1 \end{pmatrix} \rightarrow Q_{jk}^{n,1} = \begin{pmatrix} 1/n & 1 \\ 1/n & 0 \end{pmatrix} \quad [A14]$$

- Truncated polynomial vectors (see equation [A1]):

$$P^n(s \oplus z) = \frac{P^n(s) - P^n(z)}{z - s} = \frac{s^2 - z^2 + 2(s-z)/n}{z - s} = -(s+z+2/n) = -(s+2/n \quad 1) \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} \quad [A15]$$

$$\Rightarrow P^n(s \oplus k) = -(s+2/n \quad 1)$$

- Derivative discontinuity matrix and vector (see equation [16]):

$$[R_{eq}^{n+1/2,1}] = [Q^{n+1,1}]^{-1} \overbrace{[B_{n+1}]^{-1} [B_n]}^{[I_2]} [Q^{n,1}] = \begin{pmatrix} 0 & n+1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1/n & 1 \\ 1/n & 0 \end{pmatrix} = \begin{pmatrix} (n+1)/n & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rightarrow R_k^{n+1/2,1}(s) = \begin{pmatrix} (n+1)/n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s^0 \\ s^1 \end{pmatrix} = \begin{pmatrix} (n+1)/n \\ s \end{pmatrix} \quad [A16]$$

- Q' functions (3 intervals, see recurrence relations [26]):

$$Q_{1 \leftarrow 1}^{i3,3}(y_3) = Q_{1 \leftarrow 1}^3(y_3) = 1/3 + y_3 \rightarrow Q_{1 \leftarrow 1}^{i3,3}(y_3 \oplus k_3) = (-1 \quad 0) \quad [A17]$$

$$Q_{1 \leftarrow 1}^{i3,2}(y_3, s_2) = Q_{1 \leftarrow 1}^{i3,3}(y_3) \sum_{k_3=0}^1 \left[\frac{Q_{1 \leftarrow 1}^{i3,3}(y_3 \oplus k_3)}{Q_{1 \leftarrow 1}^{i3,3}(y_3)} - \frac{P^3(y_3 \oplus k_3)}{P^3(y_3)} \right] R_{k_3}^{2+1/2,1}(s_2) =$$

$$= \left[(-1 \quad 0) + \frac{1/3 + y_3}{P^3(y_3)} (y_3 + 2/3 \quad 1) \right] \begin{pmatrix} 3/2 \\ s_2 \end{pmatrix} = \frac{[-7/6] + [1/3]s_2 + [1/2]y_3 + s_2 y_3}{y_3^2 + 2/3 y_3 + 1} \quad [A18]$$

$$\rightarrow Q_{1 \leftarrow 1}^{i3,2}(y_3, s_2 \oplus s_1) = \frac{Q_{1 \leftarrow 1}^{i3,2}(y_3, s_1) - Q_{1 \leftarrow 1}^{i3,2}(y_3, s_2)}{s_2 - s_1} = \frac{-1/3 - y_3}{y_3^2 + 2/3 y_3 + 1} = \underbrace{\frac{-1/3 - y_3}{y_3^2 + 2/3 y_3 + 1}}_{Q_{1 \leftarrow 1}^{i3,2}(y_3, s_2 \oplus k_2)} (1 \quad 0) \begin{pmatrix} s_1^0 \\ s_1^1 \end{pmatrix}$$

$$Q_{1 \leftarrow 1}^{i3,1}(y_3, s_2, s_1) = Q_{1 \leftarrow 1}^{i3,2}(y_3, s_2) \sum_{k_2=0}^1 \left[\frac{Q_{1 \leftarrow 1}^{i3,2}(y_3, s_2 \oplus k_2)}{Q_{1 \leftarrow 1}^{i3,2}(y_3, s_2)} - \frac{P^2(s_2 \oplus k_2)}{P^2(s_2)} \right] R_{k_2}^{1+1/2,1}(s_1) \quad [A19]$$

and so on.

- $u_{n,i}$ inputs (3 intervals, see equation [6]):

$$\tilde{u}_{m,1}(s_m \oplus sT_{m-1}) \rightarrow \begin{cases} \tilde{u}_{1,1}(s_1) = L[u_{1,1}(\tau_1)] = \frac{1}{s_1} \\ \tilde{u}_{2,1}(s_2 \oplus s_1) = L[u_{2,1}(\tau_2)] \frac{1}{s_1} = 0 \\ \tilde{u}_{3,1}(y_3 \oplus sT_2) = L[u_{3,1}(\tau_3)] \frac{1}{s_2} \frac{1}{s_1} = \frac{1}{y_3} \frac{1}{s_2} \frac{1}{s_1} \end{cases} \quad [A20]$$

Proceeding as showed in Figure 2, computing each element as explained in this Appendix, it is possible to obtain the total process variables for the whole time span, with the advantage of projecting over specific pairs i,j of input-output variables.