The Quaternary Complex Hadamard Conjecture Of Order 2^n

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Abstract: In this paper, a complete construction of quaternary complex Hadamard matrices of order 2^n is obtained using the method of Sylvester construction and Williamson construction. Williamson construction has been generalized to obtain any kind of Hadamard matrices (Complex or Real Numbers). Non-equivalent family of Hadamard Matrices can be obtained using the Generalized Williamson construction. Finally we have discussed the Hadamard design in equivalent family of Hadamard matrices and non- equivalent family of hadamard matrices using the MATLAB.
As a future work we hope to construct some skew-complex Hadamard matrices using the above construction method.
Keywords: Hadamard matrix, complex Hadamard matrix, Sylvester construction, Williamson construction.

I. INTRODUCTION

Hadamard matrices are squarematrices with entries $\pm\mathbf{1}$ for whichsatisfies the condition $HH^T = nI_n$ where I_n is the identity matrix of order $\boldsymbol{n}$. Throughout this paper, H^T is denoted by transpose of H and any two distinct rows (or columns) of these matrices are orthogonal. Mathematically, the inner product of any two pairs of distinct rows in these matrices is $\mathbf{0}$. Moreover, The determinant of aHadamard matrix has been found to be $\pm\boldsymbol{n}^{\frac{\mathbf{n}}{\mathbf{2}}}$ where, $\boldsymbol{n}$ is the dimension of the Hadamardmatrix.[**3**]

Complex Hadamardmatrix isanother generalization of usual Hadamardmatrix H whose entries are $\{\pm1, \pm i\}$satisfying the condition $H\overline{H}^T = nI_n$ where I_n is theidentity matrix of order n and $\overline{H}^T$ is the conjugate transpose of H.It can be seen that the row/column operations on aHadamard matrix preservesallHadamard properties.Furthermore, Hadamard Matrices$\mathrm{H_1}$ and $\mathrm{H_2}$ are said to be equivalent, if $\mathrm{H_2}$ can be obtained from row/column operations on$\mathrm{H_1}$[9].

A complex Hadamard matrix H of order n is of Butsontype, if H is composed from some qth roots of unity and it is denotedby$\mathrm{BH(q;\ n)}$. For example, $\mathrm{BH(2, n)}$is denoted by real Hadamard Matrices.The elements of $\mathrm{BH(2, n)}$ are the roots of this equation$\mathrm{x^2 - 1 = 0}$ which are1 and -1 only. A $\mathrm{BH(4, n)}$is a quaternary complex Hadamard Matrix and the elements of $\mathrm{BH(4, n)}$ are the roots of this equation $\mathrm{x^4 - 1 = 0}$ which are $1, -1, \mathrm{i}, -\mathrm{i}$ [1]

Recently, there is a huge interest in $\mathrm{BH(q, n)}$ matrices. Thisis the starting point of inventing the parametric family of complex Hadamard Matrices. P.Dit.et al, in his paper, has introducedsome unknown parametric families of Complex Hadamard matrices. The existence of $\mathrm{BH(q, n)}$ is a still open problem.The simplest case is when $\mathrm{q = 2}$ which is also undecided. Real Hadamard matrices $\mathrm{BH(2, n)}$have been completely constructed up to order 28. [1]

A complete Classification of Quaternary Complex Hadamard Matrices of order 10, 12, and 14 have been presented in [1].This explains aparameterization scheme of order 10 and 12, and it has identified non-existence of parameterization scheme for order 14. [6]

This paper presents the construction ofComplex Hadamard matrices of order 2^nusing Kronecker Product Construction method and Williamson Construction method. These construction methods have been automated using MATLAB.Some complex Hadamarddesigns havealso been obtained from the proposed construction methods. The following section describes the definitions and the notations that have been used in this work.

Definition 1: (Sylvester construction)[3]

Thesimplestconstruction method is Sylvester Construction method. The first example for a Hadamard matrix was given by Sylvester in 1967. Further prove that if H is a Hadamard matrix, so is $\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$.

For example, if

$H_1 = [1]$ and $H_2 = \begin{bmatrix} H_1 & H_1 \\ H_1 & -H_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ then, $H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ **Definition**

2 : **(Kronecker Product**)[3]

Let A= $(a_{ij})_{m\times n} \in \mathbb{R}^{m\times n}$, B $\in \mathbb{R}^{p\times q}$. Then Kronecker product (or tensor product) of A and B is defined as the matrix

$$A\otimes B = \begin{pmatrix} a_{11}B \dots\dots & a_{1n}B \\ a_{m1}B \dots\dots & a_{mn}B \end{pmatrix} \in \mathbb{R}^{mp\times nq}$$

This operation is true when A and B are ComplexHadamard matrices.

Example

Let A $= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and B $= \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$ then

$$A\otimes B = \begin{pmatrix} B & 2B & 3B \\ 3B & 2B & B \end{pmatrix} = \begin{pmatrix} 2 & 1 & 4 & 2 & 6 & 3 \\ 2 & 3 & 4 & 6 & 6 & 9 \\ 6 & 3 & 4 & 2 & 2 & 1 \\ 6 & 9 & 4 & 6 & 2 & 3 \end{pmatrix}$$

The smallest Quaternary complex Hadamard matrix isof order 2,

$$C_2 = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

A quaternary complex Hadamard matrix of the next smallest order is 6,

$$C_6 = \begin{pmatrix} i & 1 & 1 & 1 & 1 & 1 \\ 1 & i & 1 & 1 & 1 & 1 \\ 1 & 1 & i & 1 & 1 & 1 \\ 1 & 1 & 1 & i & 1 & 1 \\ 1 & 1 & 1 & 1 & i & 1 \\ 1 & 1 & 1 & 1 & 1 & i \end{pmatrix}$$

The aim of this paper is to continue that work, and give a complete classification of the $BH(4,n)$ matrices up to orders 2^n.In this paper we have used several methods to develop some Hadamard matrices such as Sylvester construction, Williamson construction method and Kronecker Product.

Lemma 1.1[3]

If there exists *(1, -1)* matrices A, B, C and D of order *n* which satisfythe condition,

$$AA^T + BB^T + CC^T + DD^T = 4nI_n.$$

then $\begin{bmatrix} A & B & C & D \\ B & -A & D & -C \\ C & -D & -A & B \\ D & C & -B & -A \end{bmatrix}$ is a Hadamard matrix of order $4n$.

This method cannot be used to construct Complex Hadamard matrices. Therefore we have generalized this lemma to construct Complex Hadamardmatrices as follows.

Lemma 1.2
If there exists (*1, -1, i, -1)* matrices A, B, C and D of order *n* which satisfythe condition,

$$A\overline{A}^T + B\overline{B}^T + C\overline{C}^T + D\overline{D}^T = 4nI_n,$$

then $\begin{bmatrix} A & B & C & D \\ B & -A & D & -C \\ C & -D & -A & B \\ D & C & -B & -A \end{bmatrix}$ is a Complex Hadamard matrix of order $4n$.Here, we have constructed any complex Hadamard matrix of order 2^n using the Sylvester construction, Kronecker product and generalized Williamson construction. The proof of lemma 1.2 is very similar to the proof of lemma 1.1. In this proof, transpose of a matrix should be replaced by conjugate transpose of this matrix.

II. METHODOLOGY

Using Sylvester construction,quaternary complex Hadamard matricescan be generated. A quaternary complex Hadamardmatrix of smallest order $n = 2$ is

$$C_2 = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}.$$

$$C_4 = \begin{bmatrix} C_2 & C_2 \\ C_2 & -C_2 \end{bmatrix} = \begin{bmatrix} 1 & -i & 1 & -i \\ 1 & i & 1 & i \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{bmatrix}$$

Now

$$\overline{C_4}^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ i & -i & i & -i \\ 1 & 1 & -1 & -1 \\ i & -i & -i & i \end{bmatrix}$$

$$\overline{C_4}^T.C_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ i & -i & i & -i \\ 1 & 1 & -1 & -1 \\ i & -i & -i & i \end{bmatrix}\begin{bmatrix} 1 & -i & 1 & -i \\ 1 & i & 1 & i \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 4\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 4I_4$$

To obtain complex Hadamard matrices of order 2^n; $n \geq 3$, we have used MATLAB programming.
For Example, the following are C_2, C_4 and C_8 .
Figure 1- MATLAB Results window

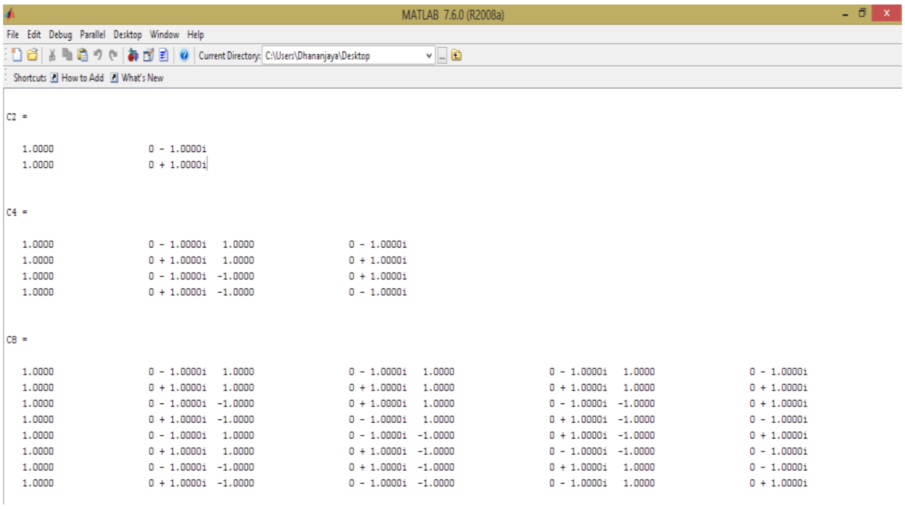


Figure 1- MATLAB Results window

Alternative method I

Also, these matrices can be obtained using Lemma 1.2.For an example, C_8 is obtained by the above lemma 1.2

Let $A = B = C = D = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$.

Clearly, $A\overline{A}^T + B\overline{B}^T + C\overline{C}^T + D\overline{D}^T = 8I_2$.

Then
$$C_8 = \begin{bmatrix} 1 & -i & 1 & -i & 1 & -i & 1 & -i \\ 1 & i & 1 & i & 1 & i & 1 & i \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & -i & -1 & i & -1 & i & 1 & -i \\ 1 & i & -1 & -i & -1 & -i & 1 & i \\ 1 & -i & 1 & -i & -1 & i & -1 & i \\ 1 & i & 1 & i & -1 & -i & -1 & -i \end{bmatrix}$$

It can be verified by considering $C_8 . \overline{C}_8^{\,T} = 8I_8$.

Alternative method II

These quaternary complex Hadamard matrices can be obtained by Kroneker Product.

For an example C_4 is obtained by Kroneker Product

$$C_2 \otimes C_2 = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \otimes \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} = \begin{pmatrix} \begin{matrix} 1 & -i \\ 1 & i \end{matrix} & -i\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \\ \begin{matrix} 1 & -i \\ 1 & i \end{matrix} & i\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & -i & -i & -1 \\ 1 & i & -i & 1 \\ 1 & -i & i & 1 \\ 1 & i & i & -1 \end{pmatrix}$$

It can be verified by considering $C_4 . \overline{C}_4^{\,T} = 8I_4$.

II.

III. RESULTS AND DISCUSSION

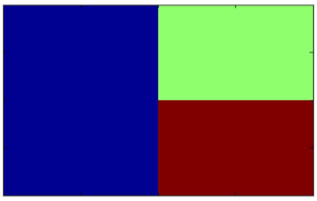
We have constructed, the complex Hadamard matrices of order 2, 4 and 8 using Kroneker Product, Sylvester Construction Method and Generalized Williamson Construction. Further Hadamard designs of those Complex Hadamard matrices are given by using MATLAB programming, by using 2 for ***i*** and 3 for –***i*** .

Figure 2

```
C2 =

     1     2
     1     3
```

Figure 2 Figure



3- C2 matrix with colors

$$C_4 = \begin{bmatrix} C_2 & C_2 \\ C_2 & -C_2 \end{bmatrix} = \begin{bmatrix} 1 & -i & 1 & -i \\ 1 & i & 1 & i \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{bmatrix}$$

$$C_8 = \begin{bmatrix} \begin{pmatrix} 1 & -i & 1 & -i \\ 1 & i & 1 & i \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{pmatrix} & \begin{pmatrix} 1 & -i & 1 & -i \\ 1 & i & 1 & i \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{pmatrix} \\ \begin{pmatrix} 1 & -i & 1 & -i \\ 1 & i & 1 & i \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{pmatrix} & -\begin{pmatrix} 1 & -i & 1 & -i \\ 1 & i & 1 & i \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 1 & -i & 1 & -i & 1 & -i & 1 & -i \\ 1 & i & 1 & i & 1 & i & 1 & i \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & -i & 1 & -i & -1 & i & -1 & i \\ 1 & i & 1 & i & -1 & -i & -1 & -i \\ 1 & -i & -1 & i & -1 & i & 1 & -i \\ 1 & i & -1 & -i & -1 & -i & 1 & i \end{bmatrix}$$

Using above notation, the above matrix can be written

```
C8 =

     1     2     1     2     1     2     1     2
     1     3     1     3     1     3     1     3
     1     2    -1     3     1     2    -1     3
     1     3    -1     2     1     3    -1     2
     1     2     1     2    -1     3    -1     3
     1     3     1     3    -1     2    -1     2
     1     2    -1     3    -1     3     1     2
     1     3    -1     2    -1     2     1     3
```

Figure 4

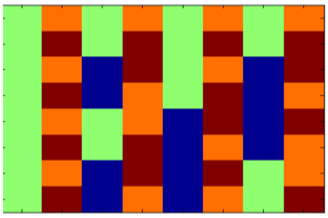


Figure 5-C_8 Matrix

Hadamard matrix of order 128

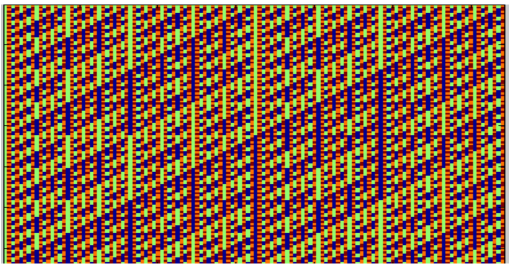


Figure 6-C_{128} Matrix

Nowapplying some row operations and column operations to the above Complex HadamardMatrices, EquivalentComplex Hadamard Matrices are obtained. For example, applying row operations to C_2, C_4, C_8 and C_{128}. We obtained the following equivalent Hdamard designs.

$C_2 C_4$ C_8 C_{128}

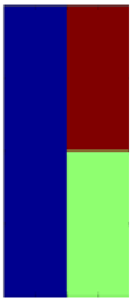


Figure 7

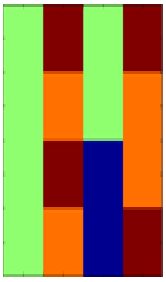


Figure 8

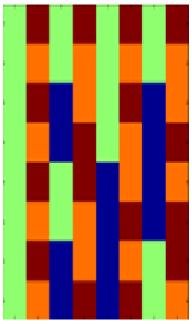


Figure 9

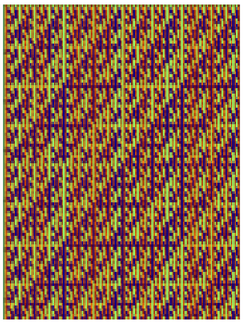


Figure 10

Similarly applying column operations, the following Hadamard designs can be obtained.

C_2 C_4 C_8 C_{128}

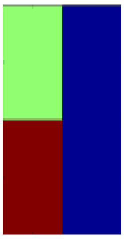


Figure 11

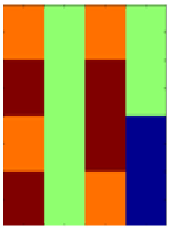


Figure 12

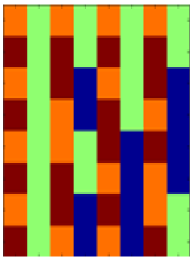


Figure 13

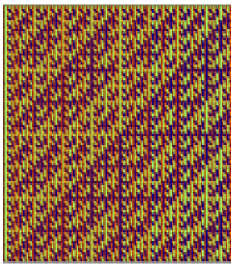


Figure 14

Further, using non-equivalent Complex Hadamard matrices, we have constructed Hadamard designs and it can be easily seen that these designs are different from the above designs. For an example, staring with below C_2which is non- equivalent to the matrix that we have discussed in previous part.

$C_2=\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}$using Hadamard Design, we have obtained

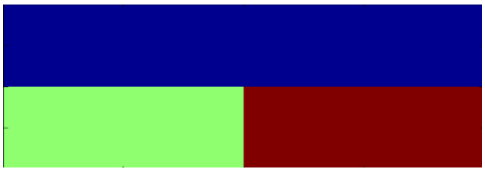


Figure 15

Then C_4, C_8, C_{128},

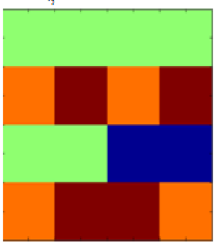


Figure 16

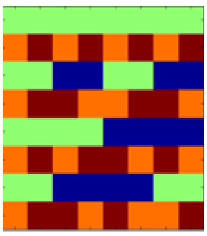


Figure 17

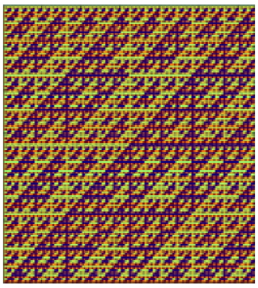


Figure 18

Applying row operations, the following designs can be obtained.

C_2

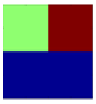


Figure 19

C_4

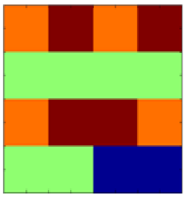


Figure 20

C_8

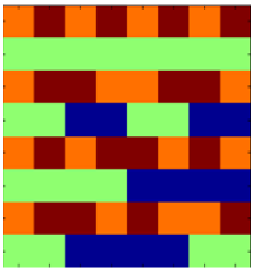


Figure 21

C_{128}

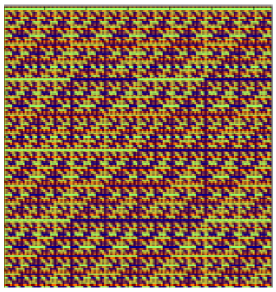


Figure 22

Similarly column operations can be applied to the above matrices.
Now, these equivalent and non-equivalent Matrices can be used to construct higher order Complex Hadamard Matrices using the method of Williamson Construction. We have several choices to A, B, C and D. Few such cases are given below.

Case I

C_8

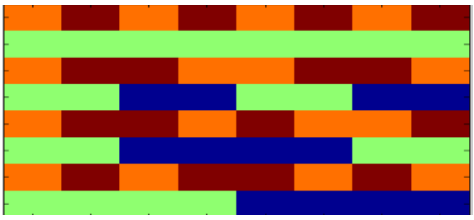


Figure 23-C_8 Matrix

It is very similar to Non-Equivalent C_8 that we have obtained in Sylvester Construction.

Case II

$$A = B = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} . C = D = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$$

C_8

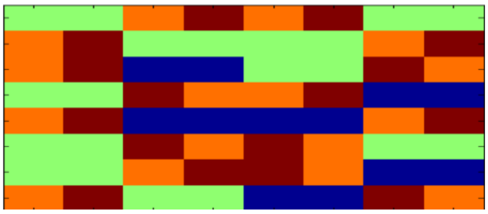


Figure 24-C_8 Matrix

This one is not similar to Non-Equivalent C_8 that we have obtained in Sylvester Construction.

Case III

$$A = B = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} . C = D = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

C_8

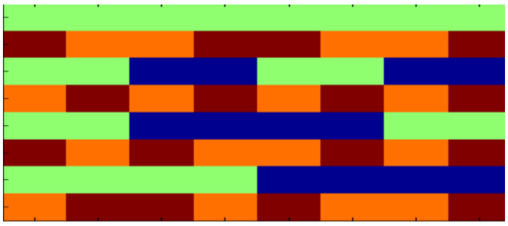


Figure 25-C_8 Matrix

Case IV

$$A = B = C = D = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}.$$

C_8

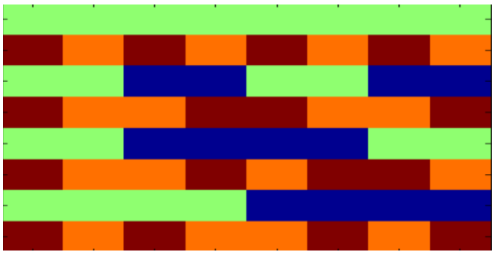


Figure 26-C_8 Matrix

Case V

$$A = B = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}. C = D = \begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}$$

C_8

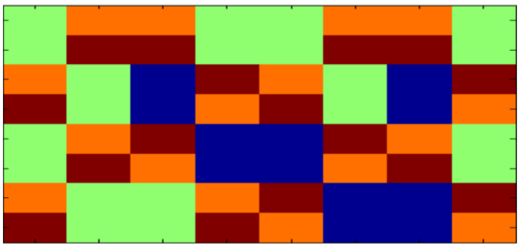


Figure 27-C_8 Matrix

Using above C_8, we have constructed C_{128}.

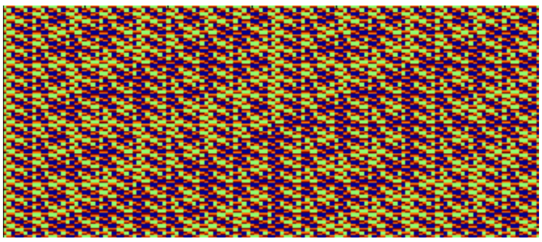


Figure 28-C128 Matrix which is obtained using above C8

IV. CONCLUSION

In this paper we have discussed Buston type Hadamard Matrix of BH(4,n).Also discussed some alternative methods to construct Complex Hadamard Matrices of order 2^n; $n \geq 3$.Further,constructed some equivalent and non- equivalent families of Complex Hadamard Matrices and their Hadamarddesigns. Finally,

obtained some non- equivalent families ofC_8, ComplexHadamard Matrices,usinggeneralized Williamson construction method.
This result can be extended to BH(q,n).

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