

# The Quaternary Complex Hadamard Conjecture of order $2^n$

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## ABSTRACT:

In this paper, a complete construction of quaternary complex Hadamard matrices of order  $2^n$  is obtained using the method of Sylvester construction and Williamson construction. Williamson construction has been generalized to obtain any kind of Hadamard matrices (Complex or Real Numbers). Non-equivalent family of Hadamard Matrices can be obtained using the Generalized Williamson construction. Finally we have discussed the Hadamard design in equivalent family of Hadamard matrices and non-equivalent family of hadamard matrices using the MATLAB.

As a future work we hope to construct some skew-complex Hadamard matrices using the above construction method.

**Keywords:** Hadamard matrix, complex Hadamard matrix, Sylvester construction, Williamson construction.

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## I. INTRODUCTION

Hadamard matrices are square matrices with entries  $\pm 1$  for which satisfies the condition  $HH^T = nI_n$  where  $I_n$  is the identity matrix of order  $n$ . Throughout this paper,  $H^T$  is denoted by transpose of  $H$  and any two distinct rows (or columns) of these matrices are orthogonal. Mathematically, the inner product of any two pairs of distinct rows in these matrices is  $0$ . Moreover, The determinant of a Hadamard matrix has been found to be  $\pm n^{\frac{n}{2}}$  where,  $n$  is the dimension of the Hadamard matrix. [3]

Complex Hadamard matrix is another generalization of usual Hadamard matrix  $H$  whose entries are  $\{\pm 1, \pm i\}$  satisfying the condition  $H\overline{H}^T = nI_n$  where  $I_n$  is the identity matrix of order  $n$  and  $\overline{H}^T$  is the conjugate transpose of  $H$ . It can be seen that the row/column operations on a Hadamard matrix preserves all Hadamard properties. Furthermore, Hadamard Matrices  $H_1$  and  $H_2$  are said to be equivalent, if  $H_2$  can be obtained from row/column operations on  $H_1$  [9].

A complex Hadamard matrix  $H$  of order  $n$  is of Butson type, if  $H$  is composed from some  $q$ th roots of unity and it is denoted by  $BH(q; n)$ . For example,  $BH(2, n)$  is denoted by real Hadamard Matrices. The elements of  $BH(2, n)$  are the roots of this equation  $x^2 - 1 = 0$  which are  $1$  and  $-1$  only. A  $BH(4, n)$  is a quaternary complex Hadamard Matrix and the elements of  $BH(4, n)$  are the roots of this equation  $x^4 - 1 = 0$  which are  $1, -1, i, -i$  [1]

Recently, there is a huge interest in  $BH(q, n)$  matrices. This is the starting point of inventing the parametric family of complex Hadamard Matrices. P. Dit et al, in his paper, has introduced some unknown parametric families of Complex Hadamard matrices. The existence of  $BH(q, n)$  is a still open problem. The simplest case is when  $q = 2$  which is also undecided. Real Hadamard matrices  $BH(2, n)$  have been completely constructed up to order 28. [1]

A complete Classification of Quaternary Complex Hadamard Matrices of order 10, 12, and 14 have been presented in [1]. This explains a parameterization scheme of order 10 and 12, and it has identified non-existence of parameterization scheme for order 14. [6]

This paper presents the construction of Complex Hadamard matrices of order  $2^n$  using Kronecker Product Construction method and Williamson Construction method. These construction methods have been automated using MATLAB. Some complex Hadamard designs have also been obtained from the proposed construction methods. The following section describes the definitions and the notations that have been used in this work.

### Definition 1: (Sylvester construction) [3]

The simplest construction method is Sylvester Construction method. The first example for a Hadamard matrix

was given by Sylvester in 1967. Further prove that if  $H$  is a Hadamard matrix, so is  $\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$ .

For example, if

$$H_1 = [1] \text{ and } H_2 = \begin{bmatrix} H_1 & H_1 \\ H_1 & -H_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \text{ then, } H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad \text{Definition}$$

**2 : (Kronecker Product)[3]**

Let  $A = (a_{ij})_{m \times n} \in \mathbb{R}^{m \times n}$ ,  $B \in \mathbb{R}^{p \times q}$ . Then Kronecker product (or tensor product) of A and B is defined as the matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \dots & \dots & \dots \\ a_{m1}B & \dots & a_{mn}B \end{pmatrix} \in \mathbb{R}^{mp \times nq}$$

This operation is true when A and B are Complex Hadamard matrices.

**Example**

Let  $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$  and  $B = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$  then

$$A \otimes B = \begin{pmatrix} B & 2B & 3B \\ 3B & 2B & B \end{pmatrix} = \begin{pmatrix} 2 & 1 & 4 & 2 & 6 & 3 \\ 2 & 3 & 4 & 6 & 6 & 9 \\ 6 & 3 & 4 & 2 & 2 & 1 \\ 6 & 9 & 4 & 6 & 2 & 3 \end{pmatrix}$$

The smallest Quaternary complex Hadamard matrix is of order 2,

$$C_2 = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

A quaternary complex Hadamard matrix of the next smallest order is 6,

$$C_6 = \begin{pmatrix} i & 1 & 1 & 1 & 1 & 1 \\ 1 & i & 1 & 1 & 1 & 1 \\ 1 & 1 & i & 1 & 1 & 1 \\ 1 & 1 & 1 & i & 1 & 1 \\ 1 & 1 & 1 & 1 & i & 1 \\ 1 & 1 & 1 & 1 & 1 & i \end{pmatrix}$$

The aim of this paper is to continue that work, and give a complete classification of the  $BH(4, n)$  matrices up to orders  $2^n$ . In this paper we have used several methods to develop some Hadamard matrices such as Sylvester construction, Williamson construction method and Kronecker Product.

**Lemma 1.1[3]**

If there exists  $(I, -I)$  matrices A, B, C and D of order  $n$  which satisfy the condition,

$$AA^T + BB^T + CC^T + DD^T = 4nI_n.$$

then  $\begin{bmatrix} A & B & C & D \\ B & -A & D & -C \\ C & -D & -A & B \\ D & C & -B & -A \end{bmatrix}$  is a Hadamard matrix of order  $4n$ .

This method cannot be used to construct Complex Hadamard matrices. Therefore we have generalized this lemma to construct Complex Hadamard matrices as follows.

**Lemma 1.2**

If there exists  $(I, -I, i, -i)$  matrices A, B, C and D of order  $n$  which satisfy the condition,

$$AA^{\bar{T}} + BB^{\bar{T}} + CC^{\bar{T}} + DD^{\bar{T}} = 4nI_n,$$

then  $\begin{bmatrix} A & B & C & D \\ B & -A & D & -C \\ C & -D & -A & B \\ D & C & -B & -A \end{bmatrix}$  is a Complex Hadamard matrix of order  $4n$ . Here, we have constructed any

complex Hadamard matrix of order  $2^n$  using the Sylvester construction, Kronecker product and generalized Williamson construction.

The proof of lemma 1.2 is very similar to the proof of lemma 1.1. In this proof, transpose of a matrix should be replaced by conjugate transpose of this matrix.

## II. METHODOLOGY

Using Sylvester construction, quaternary complex Hadamard matrices can be generated. A quaternary complex Hadamard matrix of smallest order  $n = 2$  is

$$C_2 = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

$$C_4 = \begin{bmatrix} C_2 & C_2 \\ C_2 & -C_2 \end{bmatrix} = \begin{bmatrix} 1 & -i & 1 & -i \\ 1 & i & 1 & i \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{bmatrix}$$

Now

$$\overline{C_4}^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ i & -i & i & -i \\ 1 & 1 & -1 & -1 \\ i & -i & -i & i \end{bmatrix}$$

$$\overline{C_4}^T \cdot C_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ i & -i & i & -i \\ 1 & 1 & -1 & -1 \\ i & -i & -i & i \end{bmatrix} \begin{bmatrix} 1 & -i & 1 & -i \\ 1 & i & 1 & i \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 4I_4$$

To obtain complex Hadamard matrices of order  $2^n$ ;  $n \geq 3$ , we have used MATLAB programming.

For Example, the following are  $C_2, C_4$  and  $C_8$ .

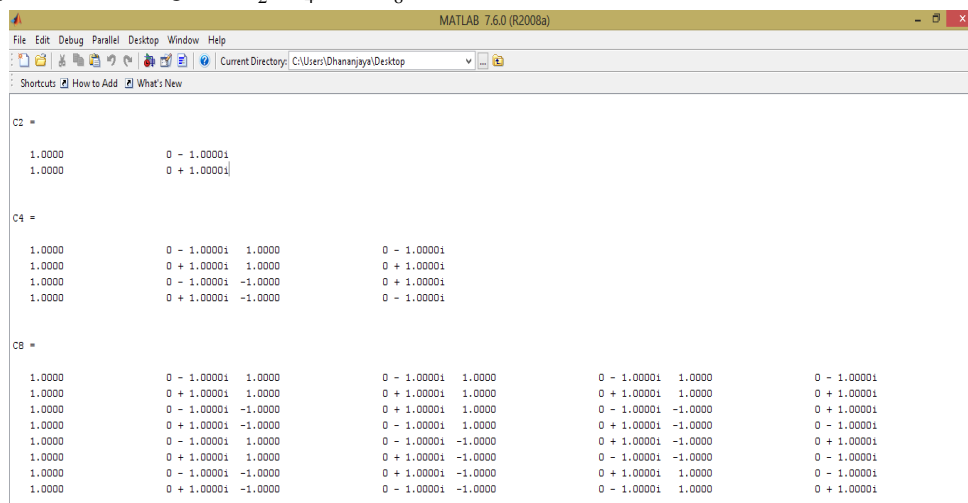


Figure 1- MATLAB Results window

Alternative method I

Also, these matrices can be obtained using Lemma 1.2. For an example,  $C_8$  is obtained by the above lemma 1.2

Let  $A = B = C = D = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$ .

Clearly,  $AA^{-T} + BB^{-T} + CC^{-T} + DD^{-T} = 8I_2$ .

Then  $C_8 = \begin{bmatrix} 1 & -i & 1 & -i & 1 & -i & 1 & -i \\ 1 & i & 1 & i & 1 & i & 1 & i \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & -i & -1 & i & -1 & i & 1 & -i \\ 1 & i & -1 & -i & -1 & -i & 1 & i \\ 1 & -i & 1 & -i & -1 & i & -1 & i \\ 1 & i & 1 & i & -1 & -i & -1 & -i \end{bmatrix}$

It can be verified by considering  $C_8 \cdot \overline{C_8}^T = 8I_8$ .

Alternative method II

These quaternary complex Hadamard matrices can be obtained by Kroneker Product.

For an example  $C_4$  is obtained by Kroneker Product

$$C_2 \otimes C_2 = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \otimes \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} = \begin{pmatrix} 1 & -i & -i(1 & -i) \\ 1 & i & i(1 & i) \end{pmatrix} = \begin{pmatrix} 1 & -i & -i & -1 \\ 1 & i & -i & 1 \\ 1 & -i & i & 1 \\ 1 & i & i & -1 \end{pmatrix}$$

It can be verified by considering  $C_4 \cdot \overline{C_4}^T = 8I_4$ .

## II. RESULTS AND DISCUSSION

We have constructed, the complex Hadamard matrices of order 2, 4 and 8 using Kroneker Product, Sylvester Construction Method and Generalized Williamson Construction. Further Hadamard designs of those Complex Hadamard matrices are given by using MATLAB programming, by using 2 for  $i$  and 3 for  $-i$ .

$C_2 =$

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Figure 2

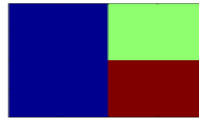


Figure 3-  $C_2$  matrix with colors

$$C_4 = \begin{bmatrix} C_2 & C_2 \\ C_2 & -C_2 \end{bmatrix} = \begin{bmatrix} 1 & -i & 1 & -i \\ 1 & i & 1 & i \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{bmatrix}$$

$$C_8 = \begin{pmatrix} \begin{pmatrix} 1 & -i & 1 & -i \\ 1 & i & 1 & i \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{pmatrix} & \begin{pmatrix} 1 & -i & 1 & -i \\ 1 & i & 1 & i \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{pmatrix} \\ \begin{pmatrix} 1 & -i & 1 & -i \\ 1 & i & 1 & i \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{pmatrix} & - \begin{pmatrix} 1 & -i & 1 & -i \\ 1 & i & 1 & i \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{pmatrix} \end{pmatrix} = \begin{bmatrix} 1 & -i & 1 & -i & 1 & -i & 1 & -i \\ 1 & i & 1 & i & 1 & i & 1 & i \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & -i & 1 & -i & -1 & i & -1 & i \\ 1 & i & 1 & i & -1 & -i & -1 & -i \\ 1 & -i & -1 & i & -1 & i & 1 & -i \\ 1 & i & -1 & -i & -1 & -i & 1 & i \end{bmatrix}$$

Using above notation, the above matrix can be written

CB =

1	2	1	2	1	2	1	2
1	3	1	3	1	3	1	3
1	2	-1	3	1	2	-1	3
1	3	-1	2	1	3	-1	2
1	2	1	2	-1	3	-1	3
1	3	1	3	-1	2	-1	2
1	2	-1	3	-1	3	1	2
1	3	-1	2	-1	2	1	3

Figure 4

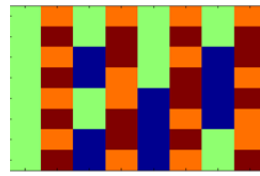


Figure 5-C<sub>8</sub> Matrix

Hadamard matrix of order 128

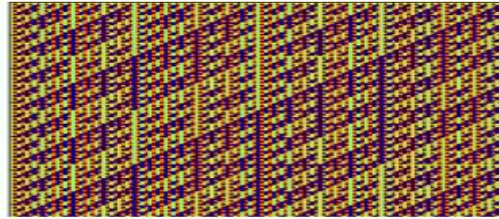


Figure 6-C<sub>128</sub> Matrix

Now applying some row operations and column operations to the above Complex Hadamard Matrices, Equivalent Complex Hadamard Matrices are obtained. For example, applying row operations to  $C_2$ ,  $C_4$ ,  $C_8$  and  $C_{128}$ . We obtained the following equivalent Hadamard designs.

$C_2C_4$



Figure 7

$C_8$

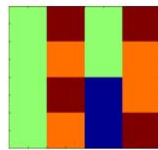


Figure 8

$C_{128}$

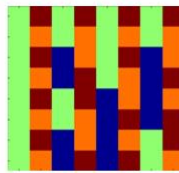


Figure 9

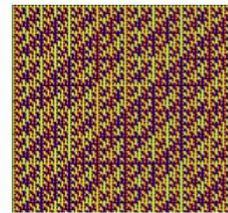


Figure 10

Similarly applying column operations, the following Hadamard designs can be obtained.

$C_2$



Figure 11

$C_4$



Figure 12

$C_8$

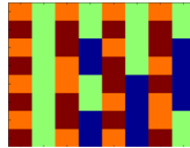


Figure 13

$C_{128}$

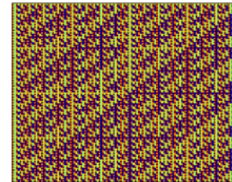


Figure 14

Further, using non-equivalent Complex Hadamard matrices, we have constructed Hadamard designs and it can be easily seen that these designs are different from the above designs. For an example, starting with below  $C_2$  which is non-equivalent to the matrix that we have discussed in previous part.

$C_2 = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$  using Hadamard Design, we have obtained

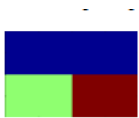


Figure 15

Then  $C_4$ ,

$C_8$ ,

$C_{128}$ ,

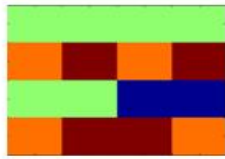


Figure 16

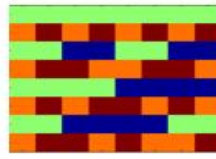


Figure 17

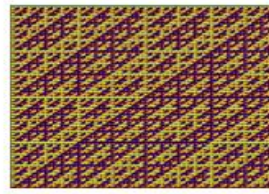


Figure 18

Applying row operations, the following designs can be obtained.

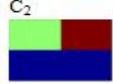


Figure 19

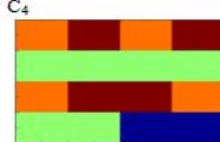


Figure 20

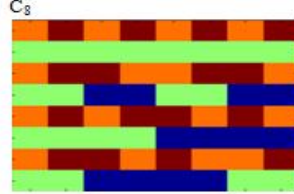


Figure 21

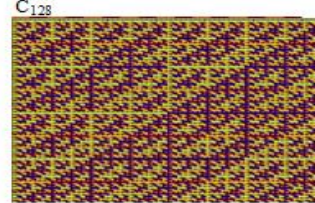


Figure 22

Similarly column operations can be applied to the above matrices. Now, these equivalent and non-equivalent Matrices can be used to construct higher order Complex Hadamard Matrices using the method of Williamson Construction. We have several choices to  $A, B, C$  and  $D$ . Few such cases are given below.

Case I

$C_8$

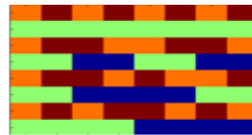


Figure 23- $C_8$  Matrix

It is very similar to Non-Sylvester Construction.

Equivalent  $C_8$  that we have obtained in

Case II

$$A = B = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}, C = D = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$$

$C_8$

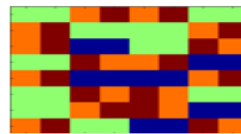


Figure 24- $C_8$  Matrix

This one is not similar to Non-Equivalent  $C_8$  that we have obtained in Sylvester Construction.

Case III

$$A = B = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}, C = D = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

$C_8$

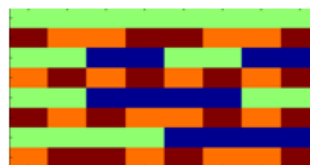


Figure 25- $C_8$  Matrix

Case IV

$$A = B = C = D = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

$C_8$

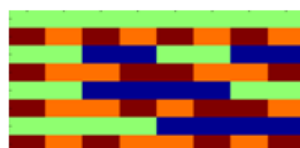
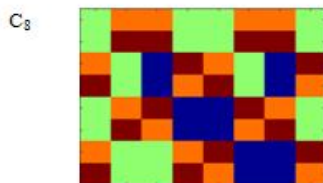


Figure 26- $C_8$  Matrix

Case V

$$A = B = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}, C = D = \begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}$$

Figure 27- $C_8$  Matrix



Using above  $C_8$ , we have constructed  $C_{128}$

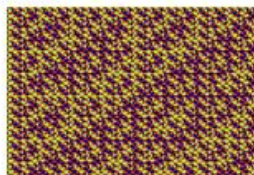


Figure 28- $C_{128}$  Matrix which is obtained using above  $C_8$

### III. CONCLUSION

In this paper we have discussed Buston type Hadamard Matrix of  $BH(4,n)$ . Also discussed some alternative methods to construct Complex Hadamard Matrices of order  $2^n$ ;  $n \geq 3$ . Further, constructed some equivalent and non-equivalent families of Complex Hadamard Matrices and their Hadamard designs. Finally, obtained some non-equivalent families of  $C_8$ , Complex Hadamard Matrices, using generalized Williamson construction method.

This result can be extended to  $BH(q,n)$ .

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